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Abstract

Asset price bubbles are a major source of macroeconomic instability, but can they play a different role in a low interest rates environment? To answer this question, I study an economy in which the natural rate of interest declines permanently, and the zero lower bound makes risk-free interest rates persistently low. Asset price bubbles redistribute wealth across generations because of the life-cycle pattern of net worth. In this way, they increase the natural interest rate by serving as a store of value for older cohorts and as a collateral for the younger ones, and the central bank can escape from the ZLB with potential output and welfare gains. Output gains are further amplified in presence of capital accumulation and the "financial accelerator". However, not all bubble types increase the natural interest rate to the same degree and, in this respect, *leveraged* bubbles have a greater impact than *unleveraged* bubbles.

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1 Introduction

Macroeconomic literature has widely studied the relationship between monetary policy and asset price bubbles from the late 1990s. At that time, the debate was about the role of monetary policy in dampening excessive fluctuations in asset prices, viewed as a potential source of macroeconomic instability and financial crises (see, e.g., Bernanke and Gertler 1999, 2001; Cecchetti et al., 2000). This old and settled debate has gained new momentum because of the boom-and-bust cycle in the US housing market of the early 2000s. However, the resulting new body of literature is developing along the old lines of research, which focused on the impact of the policy rate on asset price bubbles (e.g., Galí 2014, 2021; Dong et al., 2020), disregarding the recent developments in the advanced economies. This paper overturns the perspective, exploring the relationship that runs from asset price bubbles to monetary policy, and showing that bubbles can affect the latter by relaxing the zero lower bound constraint in a low interest rates environment.

In the last decades, advanced economies have gone through a dramatic downward adjustment in the risk-free real and nominal interest rates because of the historical declining pattern of the "natural" interest rate, which has reached extraordinary low/negative levels.[1] As a consequence, low interest rates have become highly persistent even after the recovery from the Great Recession, exposing the economy to more frequent zero lower bound (ZLB) episodes and thus deeper and longer recessions than usual (Kiley and Roberts, 2017). Against this backdrop, one aspect of the relationship between monetary policy and asset price bubbles becomes relevant: the potential impact of bubbles on the nominal interest rate.

The reason is straightforward. Wealth gains from asset price bubbles are unevenly distributed across generations due to the life-cycle pattern of assets and debt. The older cohorts with large asset holdings experience a massive increase in net worth that discourages saving. On the other hand, younger cohorts have a low stock of rising wealth, but they are more prone to borrow and can take on higher debt via appreciated collaterals.[2] As asset price bubbles redistribute wealth across generations by serving as a store of value and collateral, they can reduce the supply of saving and foster the demand for borrowing, absorbing the excess of saving underlying a low/negative natural interest rate. The relevance of this theoretical mechanism at the ZLB is the main object of the

[1] Although the natural interest rate consistent with the potential output and stable prices (Wicksell, 1898) is not observable, the estimates of Laubach and Williams (2016) for the US show that its downward trend started from the 1980s. Holston et al. (2017) find a similar decline for other industrialized economies (Canada, the Eurozone and, the UK). As these estimates refer to a medium-term natural interest rate, they point to a negative value only in the case of the Eurozone. Nevertheless, other empirical evidence (e.g., Cúrdia, 2015) supports a short-term negative natural interest rate.

[2] Bonchi (2020) discusses in detail the empirical evidence supporting this view.

present paper, which aims to show that *redistributive* bubbles can affect output in a low interest rates environment, unlike in normal times.

To that aim, I develop a stylized three-period overlapping generations (OLG) model that features asset price bubbles and non-neutral monetary policy. The benchmark model accounts for the life-cycle behavior of saving and net worth, and thus the natural interest rate is endogenously determined in the credit/risk-free assets market. Asset price bubbles, formalized as intrinsically worthless assets, emerge rationally in the OLG economy and play the twofold role of a store of value and ("bubbly") collateral through which they can influence the natural interest rate. The effect of bubbles going through the natural rate of interest has radically different implications for macroeconomic outcomes and welfare, whether the ZLB is binding or not.

When the natural interest rate is not negative, the economy lies in a steady state equilibrium with output at the potential and inflation at the target because the *ZLB is not binding*. The emergence of bubbles raises the natural interest rate along the transition to the new bubbly equilibrium, but this results in higher interest rates without altering output. On the other hand, a higher natural interest rate underlies a redistribution from young to old age without affecting the middle generation. Despite the bubbly collateral, bubbles crowd out credit, reducing the resources available for the borrowing-constrained young households. Equally, they raise the wealth in old age, boosting the consumption of the elderly. According to the model calibrations, this redistribution is welfare-reducing, given the greater weight to the young-age consumption losses in the lifetime utility.

In contrast, when the natural interest rate falls deeply in negative territory, the *ZLB is binding*, and output and inflation gaps are negative in a bubbleless economy. If the upward pressure of bubbles drives the natural rate into non-negative/positive territory, the central bank can escape from the ZLB with resulting output gains. In this case, the model calibrations point to a welfare-enhancing bubble because the output increase positively affects the consumption/income in middle age, though bubbles still redistribute resources from the young to the old generation.

While the benchmark model points to the effect of redistributive bubbles on aggregate demand working through the natural interest rate, amplification mechanisms could interact with this channel. For example, the introduction of capital and endogenous fundamental collateral in the model shows that the "financial accelerator" can amplify the impact of bubbles on output greatly, not only closing the initial negative output gap but also fostering the potential output through capital accumulation. Indeed, the "financial accelerator" enlarges the initial output gains from the bubble emergence. This, in turn, determines an even higher income for the middle generation that can now both consume and save more, providing additional credit to the borrowing-constrained young households that accumulate capital and consume by pledging bubbly and fundamental collaterals. Hence, according to the extended model calibration, the stock of capital rises, along with the young-age consumption, and all generations are now better off in the final bubbly equilibrium.

Finally, I augment my benchmark model with an exogenous probability of bubble bursting and an incomplete credit market, accounting for the difference between leveraged and unleveraged bubbles. The distinction between the two bubble types helps to clarify the bubble's role of primarily influencing the natural interest rate. An unleveraged bubble serves only as a store of value, unlike a leveraged one used as collateral. While an *unleveraged* bubble raises the natural rate of interest, although it does not drive it into positive territory leaving the central bank stuck at the ZLB, a *leveraged* bubble delivers a non-negative/positive natural interest rate, and the monetary authority can escape from the ZLB with consequently higher output gains.

The rest of the paper is structured as follows. Section 2 presents the related literature. I spell out the benchmark model with rational bubbles and monetary policy in Section 3, and I study its steady state, along with the transition to a bubbly equilibrium, in Section 4. Section 5 provides the two extensions of the benchmark model. Section 6 concludes the paper.

2 Related Literature

My paper builds on the models of rational bubbles, in which bubbly assets emerge to mitigate some inefficiencies in the financial market, such as a shortage of investment opportunities (e.g., Samuelson, 1958; Tirole, 1985) and credit market frictions (e.g., Martin and Ventura, 2011, 2012; Fahri and Tirole, 2012; Hanson and Phan, 2017; Hirano and Yanagawa, 2017; Miao and Wang, 2018; Bengui and Phan, 2018). However, my framework differs from these models in the introduction of non-neutral monetary policy and its central role. In this respect, the papers closest to mine are Galí (2014, 2021), Miao et al. (2019), Biswas et al. (2020), and Asriyan et al. (2021). While Miao et al. (2019) focus on the different effects of "leaning against the wind" policies in stable and unstable bubbly steady states, I study the different effects of the transition to a stable bubbly steady state whether the original bubbleless economy is or not at the ZLB. Galí (2014, 2021) formalizes monetary policy in terms of an interest rate rule, but he investigates the traditional relationship that runs from the policy rate to asset price bubbles without accounting for the relationship running in the opposite direction at the ZLB. My paper also differs from that of Asriyan et al. (2021), who investigate the role of the central bank as supplier of assets. In their framework without nominal rigidities, expanding the supply of unbacked assets (money) in a liquidity trap redistributes resources from *savers* to *old* agents, crowding out inefficient investments and raising the consumption of all generations. I abstract from the role of the central bank as supplier of assets, and I study the effect of an increase in unbacked assets (bubbles) when the ZLB is binding, and downwardly rigid wages cause output losses. In this case, the bubble is welfare-enhancing because the income/consumption of the middle-aged households (*savers*) increase, even if resources are redistributed from *young* to *old* agents. Finally, Biswas et al. (2020) also investigate the potential output gains from bubbles through a model featuring binding ZLB and downward (real) wage rigidity. However, they do not introduce the possibility to default, which allows for risk-shifting in my work highlighting the differentiated output gains from leveraged and unleveraged bubbles.

My work also relates to the recent extensive literature on secular stagnation. Unlike the traditional literature on liquidity traps that studies ZLB episodes caused by temporary shocks (e.g., Krugman, 1998; Eggertsson and Woodford, 2003) or self-fulfilling expectations (e.g., Behnabib et al., 2001), the new secular stagnation theory views recent long-lasting ZLB episodes as a result of the structural decline in the natural interest rate driven by slow-moving forces (Summers, 2014; Baldwin and Teulings, 2014; Gordon, 2015; Eggertsson et al., 2016; Eggertsson et al., 2019; Bacchetta et al., 2020). I stick to this theory for the interpretation of the current low interest rates environment. In particular, I enrich the theoretical model of Eggertsson et al. (2019) with rational asset bubbles to emphasize a special role for them in such an environment. Bacchetta et al. (2020) also find that bubbles can bring the economy out of the ZLB, but bubbles (or money) crowds out capital with a detrimental effect on the supply side in the long-run. This is because they abstract from bubbly collateral ("crowd in" effect) and the interaction between the demand side and the supply side through the "financial accelerator". In my extended model with capital, bubbles can be used as collaterals, and they increase the aggregate demand by relaxing the ZLB, determining a further appreciation of the (fundamental) collaterals that foster consumption and capital accumulation with a beneficial effect on the demand side and the supply side in the long-run.

Finally, the present paper is connected to the literature on the macroeconomic effects of households' heterogeneity in the marginal propensity to consume (MPC) (e.g., Galí et al. 2007; Bilbiie 2008, 2021; Auclert and Rognlie, 2018). The life-cycle structure of my model makes the different generations heterogeneous in terms of MPC. However, the reallocation from young to old age does not directly affect equilibrium output through the net effect on aggregate demand but indirectly through the natural interest rate and the ZLB. Bubbles redistribute resources between generations with the same MPC (equal to one), and this is precisely why their emergence does not alter output/demand for a positive policy rate. Instead, at the ZLB, the positive effect on output/demand comes from the higher natural interest rate reflecting the intergenerational redistribution and taking the economy out of the ZLB.

3 The Benchmark Model

I consider a three-period OLG economy without capital. The generation born at t is composed of N_t agents, and the constant ratio between the size of the young and middle generations is $(1+g) = \frac{N_t}{N_{t-1}}$, where g is also the population growth rate. The economy consists of three agents who form expectations rationally and who are perfectly informed: households, firms and a central bank in charge of monetary policy; and two markets for financial assets: the credit and bubbles markets.

Agents live for three periods: when young, they are borrowers; in their middle age, they save for retirement; once retired, they consume all the proceeds from their savings. Middle-aged households run firms, whose real profits are Z_t,[3] and supply their labor endowment $\bar{L}$ inelastically for a wage W_t. However, their labor income depends on the labor demand from firms, L_t, because the model incorporates the labor rationing approach (Schmitt-Grohé and Uribe, 2016). Indeed, labor and goods markets are perfectly competitive, but there exists a downward nominal wage rigidity (DNWR) that makes monetary policy non-neutral. When this rigidity is at work, the labor demand is lower than the supply, with a consequent lower aggregate output/income for middle-aged households.[4] As only the middle generation gets a positive income, it supplies funds to the young one in exchange for bonds and to the old one in exchange for bubbly assets.

I describe the behavior of the agents and the functioning of the asset markets in this section.

3.1 Households

When young, households borrow by issuing a one-period risk-free bond. Middle-aged households, which earn the income $Y_t = \frac{W_t}{P_t} L_t + Z_t$, can invest their savings in bonds or buy different varieties

[3] I abstract from the equity market to keep the model simple. Its presence does not substantially alter the model (see Appendix A.2), but it clarifies the source of the new (old) bubbles, which are those attached to the shares of new (old) firms, and that of the bubble destruction process, which regards the bubbly assets attached to the old firms exiting the market.

[4] The structure of the model does not change with an alternative nominal friction such as Calvo pricing (Calvo, 1983). Eggertsson et al. (2019) derive this result in a similar setting. However, the DNWR allows for replicating the output losses associated with ZLB episodes, keeping the model more tractable. For a discussion of the empirical and theoretical relevance of downwardly rigid wages, I refer to Schmitt-Grohé and Uribe (2016) and Eggertsson et al. (2019).

of bubbles from the old generation. In each period, middle-aged households create $\delta \in (0,1)$ units of a variety of "bubble", whose price is $P^B_{t|t} \geq 0$, though the bubbly asset is intrinsically worthless. The bubble is a claim on future savings because it entitles the owner to receive a payment from the next generation. Middle-aged households issue this claim directly by initiating a new bubble, while they buy the claims issued by the past generations by purchasing old bubbles. The quantity of bubbly assets grows at the same rate as the population, but a fraction δ of old bubbly assets is destroyed in each period. Bubbles (both new and old ones) allow agents to carry over funds to old age, but new bubbles also improve the ability to repay debt in middle age. Consequently, young households, whose borrowing capacity is constrained by the exogenous fundamental collateral D, can demand more funds in the credit market by using the future bubble as a bubbly collateral.

The representative household solves the maximization problem:

$$\max_{C^m_{t+1}, C^o_{t+2}, Q^B_{t+1|t+1-j}} E_t \left\{ \ln C^y_t + \beta \ln C^m_{t+1} + \beta^2 \ln C^o_{t+2} \right\}$$

s.t.

$$C^y_t = B^y_t \tag{1}$$

$$C^m_{t+1} = Y_{t+1} + \delta P^B_{t+1|t+1} - (1+r_t) B^y_t - B^m_{t+1} - \sum_{j=0}^{\infty} P^B_{t+1|t+1-j} Q^B_{t+1|t+1-j} \tag{2}$$

$$C^o_{t+2} = (1+r_{t+1}) B^m_{t+1} + (1-\delta)(1+g) \sum_{j=0}^{\infty} P^B_{t+2|t+1-j} Q^B_{t+1|t+1-j} \tag{3}$$

$$(1+r_t) B^y_t = D + \delta E_t P^B_{t+1|t+1}. \tag{4}$$

The household's utility, which is discounted at the rate β, is given by the real consumption of each generation, C^i_t with $i = y, m, o$. B^y_t and B^m_t denote respectively the real value of bonds issued by the young generation and bought by the middle one. The (gross) rate of return on bonds is $(1+r_t)$. $Q^B_{t|t-j}$ and $P^B_{t|t-j}$ are the quantity at time t of the bubbly asset introduced by the cohort $t-j$ and its price. $Q^B_{t|t-j} P^B_{t|t-j}$ is accordingly the expenditure for the bubbly asset $t-j$ from the middle generation, whose total expenditure for all the varieties of bubbles is given by the summation in equation (2). Equation (4) is the debt limit that is binding by assumption for young households.[5]

[5] This holds for $D < [Y_t - \beta(1+\beta)\delta P^B_{t|t}]/[1+\beta(1+\beta)]$, a condition that is met in all the simulations presented in the paper.

The optimality conditions for the household's problem are the standard Euler equation

$$\frac{1}{C_t^m} = \beta\left(1 + r_t\right) E_t \frac{1}{C_{t+1}^o} \tag{5}$$

and

$$P_{t|t-j}^B = \left(1 - \delta\right)\left(1 + g\right) \beta E_t \left[\left(\frac{C_t^m}{C_{t+1}^o}\right) P_{t+1|t-j}^B\right], \tag{6}$$

which expresses the price of the bubbly asset. As the bubble has no fundamental value, it is valued if the household expects to gain profit by selling it, and thus its current price depends on the discounted expected price in the next period.

3.2 Firms

Firms operate for just one period and, as their number is equal to the size of the middle generation, the economy's growth rate is g. The production technology is

$$Y_t = L_t^{\alpha}, \tag{7}$$

where $0 < \alpha < 1$. Taking the price of goods (P_t) and that of labor (W_t) as given, firms maximize their real profit

$$Z_t = Y_t - \frac{W_t}{P_t} L_t \tag{8}$$

subject to (7). The resulting optimality condition is the labor demand

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1}. \tag{9}$$

Workers are unwilling to accept nominal wages lower than a minimum level (Schmitt-Grohé and Uribe, 2016), and the resulting DNWR is

$$W_t = \max\left(\gamma \Pi^* W_{t-1}, P_t \alpha \bar{L}^{\alpha-1}\right), \tag{10}$$

where $\gamma \in (0, 1]$ and $\gamma\Pi^* \geq 1$. The first term in the max operator denotes the minimum wage level, which is a fraction γ of the past nominal wage indexed to the gross inflation target $\Pi^* \geq 1$, while the second term is the "flexible" wage level compatible with full employment. When labor market clearing requires an increase in W_t from the last period by at least $\gamma\Pi^*$, the nominal wage is flexible, and there is full employment ($L_t = \bar{L}$). When the wage should increase by less than $\gamma\Pi^*$ to maintain the full employment of resources, the downward wage rigidity prevents such price adjustment, and involuntary unemployment arises ($L_t < \bar{L}$).

3.3 The Central Bank

The central bank behaves according to the interest rate rule

$$1 + i_t = \max\left[1, \left(1 + r_t^f\right)\Pi^*\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right], \tag{11}$$

where $\phi_\pi > 1$ and the gross inflation rate is given by $\Pi_t = P_t/P_{t-1}$. The central bank maneuvers the nominal interest rate i_t to track the natural interest rate r_t^f, which is consistent with the potential output ($Y^f = \bar{L}^\alpha$) and inflation at the targeted level Π^* (Cúrdia et al., 2015). However, if the natural interest rate turns negative ($1 + r_t^f < 1$) and the targeted inflation rate is not sufficiently high, the central bank would set a negative nominal interest rate, but it cannot because of the ZLB in equation (11), $1 + i_t \geq 1$.[6] Finally, the gross real interest rate has to satisfy the Fisher condition

$$1 + r_t = (1 + i_t)\, E_t \Pi_{t+1}^{-1}, \tag{12}$$

where E_t denotes the expectation operator.

3.4 Credit and Bubbles Markets

Bubbly assets market clearing requires

$$Q_{t|t-j}^B = \delta\,(1-\delta)^j\,. \tag{13}$$

The demand for the bubble variety $t - j$ at time t has to be equal to its supply, which depends on the quantity created in $t - j$ and on that which survives over time. Given the assumption on the endowment of new bubbles and those on destruction and growth rates of old bubbles, the total

[6]Despite the drawbacks of a cashless economy, the use of an interest rate rule is coherent with the aim of this paper, which investigates the effect of bubbles when the ZLB and the DNWR bind simultaneously, determining output losses. Furthermore, I implicitly assume that unconventional monetary policies such as "quantitative easing" or "helicopter money", even if possible, cannot fully absorb the excess of savings, leaving room for the bubble to emerge.

amount of bubbles in the economy is equal to the size of the middle generation.[7] The economy's bubble index, which includes new and old bubbles, is

$$P_t^B = \frac{1}{N_{t-1}} \delta \sum_{j=0}^{\infty} (1-\delta)^j \, P_{t|t-j}^B, \tag{14}$$

while the index for old bubbles only is

$$B_t = \frac{1}{N_{t-1}} \delta \sum_{j=1}^{\infty} (1-\delta)^j \, P_{t|t-j}^B. \tag{15}$$

Both indexes are normalized in terms of the size of the middle generation, N_{t-1}. The equation for the aggregate bubble index can be rewritten as

$$P_t^B = U_t + B_t = (1+g) \left[\frac{B_{t+1}}{(1+r_t)} \right] \tag{16}$$

by using equations (5), (6), (14) and (15). By assuming perfect foresight, I disregard here and afterwards the expectation term. Equation (16), in which $U_t = \delta P_{t|t}^B / N_{t-1}$ denotes the value of the new bubbly assets, is a no-arbitrage condition. New and old bubbly assets will be valued from rational agents in the next period if their expected rate of return is equal to that of bonds, that is, the real interest rate. The term $(1+g)$ undoes the effect of the growth in the quantity of bubbles on their total value, which depends accordingly only on their price.[8]

The equilibrium in the credit market requires

$$(1+g) \, B_t^y = B_t^m. \tag{17}$$

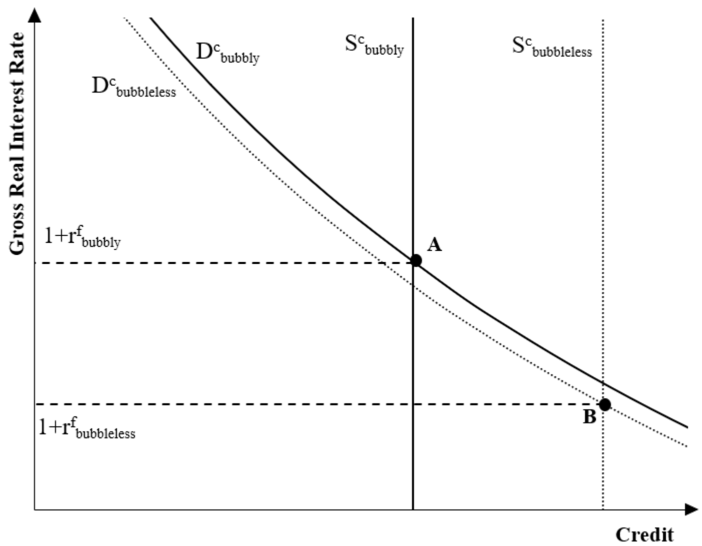
Denote credit demand from young households with D_t^c and the credit supply of middle-aged households with S_t^c. Plugging (4) into credit demand obtains

$$D_t^c = \left(\frac{1+g}{1+r_t} \right) (D + U_{t+1}) \, . \tag{18}$$

[7] The growth rate of the quantity of bubbly assets and the size of middle and old generations do not appear in equation (13), because they cancel each other out. Given (13), the aggregate supply of bubbles is $N_{t-1} Q_t^B = N_{t-1} \sum_{j=0}^{\infty} Q_{t|t-j}^B = N_{t-1}$, which means that each middle-aged household owns one unit of bubbles overall.

[8] I neutralize fully the effect of growing bubbles quantity, which implies one unit of bubbles for each middle-aged household, by normalizing the bubble index in terms of the aggregate bubble supply, N_{t-1}. In general, the assumption of growing bubbles quantity does not affect my results, as I prove formally in Appendix A.1. There, I provide a modified version of the model with zero population growth, a fixed unit supply of bubbly assets in aggregate, and the economy still growing at a positive rate because of productivity growth.

Figure 1: Equilibrium in the credit market



Combining (2), (3), (4), (5) and (16) yields the credit supply,[9]

$$S_t^c = \frac{\beta}{1+\beta}\left(Y_t - D - U_t - B_t\right) - \frac{1}{1+\beta}\left(B_t + U_t\right). \tag{19}$$

Equations (18) and (19) describe the twofold role of the bubble in the economy. First, bubbles serve as a store of value, reducing the supply of saving in the credit market (19). An alternative investment vehicle diverts resources, B_t, away from risk-free bonds, and it induces agents to save less by providing an additional income, $B_t + U_t$, in old age. Second, future bubbly assets U_{t+1} serve as a collateral fostering demand for borrowing in (18). Higher demand for credit from young households results in a higher debt to repay for middle-aged ones, and the higher debt U_t decreases the credit supply in (19). We can derive the real interest rate that clears the credit market,

$$(1 + r_t) = (1 + g)\left[\frac{(1+\beta)\left(D + U_{t+1}\right)}{\beta\left(Y_t - D - U_t - B_t\right) - \left(B_t + U_t\right)}\right], \tag{20}$$

by equating (18) and (19). The equilibrium real rate corresponds to the natural interest rate at the potential level of production, $Y_t = Y^f$.

The two channels through which the bubble alters the natural rate of interest are depicted graphi-

[9] We derive $\sum_{j=0}^{\infty} P^B_{t|t-j} Q^B_{t|t-j} = U_t + B_t$ and $(1-\delta)\sum_{j=0}^{\infty} P^B_{t+1|t-j} Q^B_{t|t-j} = B_{t+1}$ from (13), (14) and (15).

cally in Figure 1. The figure plots the credit demand and supply curves both in a bubbleless economy, in which $B_t = U_t = 0$, and in a bubbly one corresponding to $B_t > 0$ and $U_t \geq 0$. Compared to the bubbleless economy, bubbly assets reduce the credit supply and foster the demand for credit. As these two effects push r_t^f up by shifting the credit supply curve left and the credit demand curve right, the natural rate of interest is higher in a bubbly economy. Therefore, it could be positive in a bubbly economy while it is negative in a bubbleless one.

4 Steady State Equilibrium

A perfect foresight equilibrium is a set of quantities $\{C_t^y, C_t^m, C_t^o, B_t^y, B_t^m, Y_t, Z_t, L_t, B_t\}$ and prices $\{P_t, W_t, r_t, i_t\}$ that solve (1), (2), (3), (4), (5), (7), (8), (9), (10), (11), (12), (16) and (17), given $\{U_t\}$ and initial values for W_{-1}, B_{-1}^m and B_{-1}. I assume $U_t = U$ with $U \geq 0$. Then, we obtain the law of motion of the old bubble by combining (16) and (20):

$$B_{t+1} = \frac{(1+\beta)(D+U)(U+B_t)}{\beta(Y_t - D - U - B_t) - (U + B_t)} = H(B_t, U). \tag{21}$$

For a given $Y_t = Y$, a *bubbleless* steady state equilibrium corresponds to the pair $(B, U) = (0, 0)$ such that $B = H(0, 0) = 0$, while a *bubbly* steady state equilibrium is a pair (B, U) satisfying $B = H(B, U)$ with $B \in (0, Y^f)$. The bubbly equilibrium exists if

$$D < \frac{\beta}{1+\beta}(Y - D), \tag{22}$$

which is a necessary and sufficient condition. An overly low debt limit prevents young households from issuing enough bonds to absorb all savings. The excess of savings pushes the real interest rate below the economy's growth rate in a bubbleless economy, and condition (22) can be rewritten as

$$(1+g)\left[\frac{(1+\beta)D}{\beta(Y-D)}\right] = 1 + r_{nb} < 1 + g, \tag{23}$$

where r_{nb} is the real interest rate when bubbles are not valued, and the subscript *nb* stands for "no-bubble". Under such circumstances, rational agents find it profitable to invest in intrinsically worthless assets (Samuelson, 1958; Tirole, 1985). More precisely, when condition (22) holds, there exists a continuum of stable $(B^S(U), U)$ and unstable $(B^U(U), U)$ bubbly equilibria for any $U \in$

$[0,\bar{U})$, where $\bar{U} = [(1+2\beta)D - \beta Y]^2 / [4\beta(1+\beta)(Y-D)]$.[10] In what follows, I restrict my attention to the stable bubbly equilibrium. In particular, the focus is on a *bubbly* full employment equilibrium in comparison with two different *bubbleless* equilibria: with and without binding ZLB.

4.1 Asset Price Bubbles and the ZLB

4.1.1 Bubbleless Economy

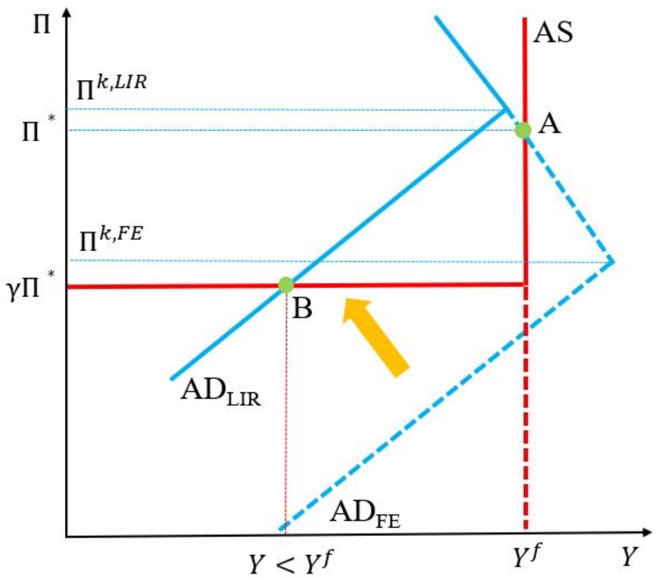
The steady state equilibrium can be expressed by aggregate demand and supply, which consist of two regimes. The DNWR determines the regime of aggregate supply (AS) through the inflation rate. If $\Pi \geq \gamma\Pi^*$, the flexible nominal wage is higher than, or at least equal to, the minimum wage in (10) and the labor market clears. AS, which can be computed from equations (7), (9) and (10), corresponds accordingly to the potential output, $Y_{AS} = \bar{L}^{\alpha} = Y^f$. On the contrary, if the flexible wage level is lower than the minimum wage imposed by the DNWR, nominal wages cannot adjust to clear the labor market. Hence, price/wage inflation is $\Pi = \gamma\Pi^*$, while output and employment are determined by demand. Figure 2 depicts the two components of the AS curve, respectively, as a vertical and a horizontal segment.

The monetary policy rule (11) governs the regime of aggregate demand (AD) through the ZLB. To save space, I report the equation(s) of the AD in Appendix B.3. If the ZLB is not binding and thus $1+i>1$, the central bank raises the policy rate more than proportionally ($\phi > 1$) in response to an inflation increase. The variation in the policy rate increases the real interest rate, contracting demand and stabilizing inflation. Hence, AD is negatively related to inflation for a positive policy rate, and a downward-sloping AD curve depicts it in Figure 2.[11] In contrast, there is a positive relationship between aggregate demand and inflation at the ZLB ($1+i=1$), and this relationship takes the shape of an upward-sloping AD curve in the same figure. The ZLB inhibits the policy rate, and thus the real interest rate depends only on the inflation level through the Fisher equation (12): $1+r = 1/\Pi$. Therefore, a surge in inflation decreases the real interest rate and expands demand. Figure 2 plots the inflation level Π^k, at which the central bank hits the ZLB, as a kink in the AD curve, and we can

[10] A formal proof for condition (22) is given in Appendix B.1, along with a graphical representation of stable and unstable equilibria, and the old bubble dynamics (Figure 7). The stability of the bubbly equilibrium depends on the condition $\partial H(B,U)/\partial B < 1$, which also guarantees the stationarity of the old bubble, as proved in Appendix B.2. The existence of the bubble, as well as its stationarity, is verified in all the simulations reported in the paper. Finally, the condition $r < g$ holds also in a bubbly equilibrium. Indeed, the price of old bubbles grows at the rate of r and, given the presence of new bubbles, the aggregate bubble would grow unboundedly if $g = r$. There is accordingly an upper bound on B as in Galí (2014): $B^U(0) = \frac{\beta}{1+\beta}(Y-D) - D$.

[11] For the sake of exposition, I follow Eggertsson et al. (2019) in plotting the downward-sloping AD curve as linear, though it is in general non-linear. Of course, my results would be the same with the general non-linear representation.

Figure 2: FE and LIR equilibria



Note: Color to be used.

compute this particular inflation level by equating the two arguments on the right-hand side of (11):

$$\Pi^k = \left[\frac{1}{(1+r^f)}\right]^{\frac{1}{\phi_\pi}} \Pi^{*\,\frac{\phi_\pi - 1}{\phi_\pi}}. \tag{24}$$

The AD kink is an indicator of the risk of a ZLB episode: for a given inflation target, the lower is Π^k, the lower is the risk of hitting the ZLB because a sufficiently high r^f gives the central bank enough space to cut the policy rate if necessary.

Figure 2 shows that two different bubbleless equilibria are possible according to the value of the natural interest rate. The first equilibrium occurs if the natural interest rate is non-negative in a bubbleless economy, $1 + r^f_{nb} \geq 1$. In this case, the central bank can steer the policy rate to keep output at the potential and inflation at the target, and the economy stays at the steady state equilibrium A, which corresponds to the curve AD_{FE} and features $r_{nb} = r^f_{nb} \geq 0$, $i \geq 0$, $Y = Y^f$ and $\Pi = \Pi^*$. I refer to this kind of equilibrium as the “full employment” (FE) equilibrium.

The second equilibrium arises, if a permanent change to r^f occurs, and its level falls deeply in negative territory so that

$$1 + r^f_{nb} < \frac{1}{\Pi^*} \leq 1. \tag{25}$$

This is the case of binding ZLB, whose resulting equilibrium features negative output and inflation gaps. When the inflation target is not sufficiently high to drive the real interest rate to its negative natural level, monetary policy is constrained by the ZLB, and it cannot keep output at the potential level and inflation at the targeted level. The AD curve moves, with the AD kink, from AD_{FE} to AD_{LIR}. Then, the new equilibrium is B, in which $r^f_{nb} < r_{nb} \leq 0$, $i = 0$, $Y < Y^f$ and $\Pi = \gamma\Pi^* \leq \Pi^*$. I refer to this equilibrium as the "low interest rates " (LIR) equilibrium because the ZLB episode delivers low nominal and real interest rates.[12] In particular, the real interest rate is negative.

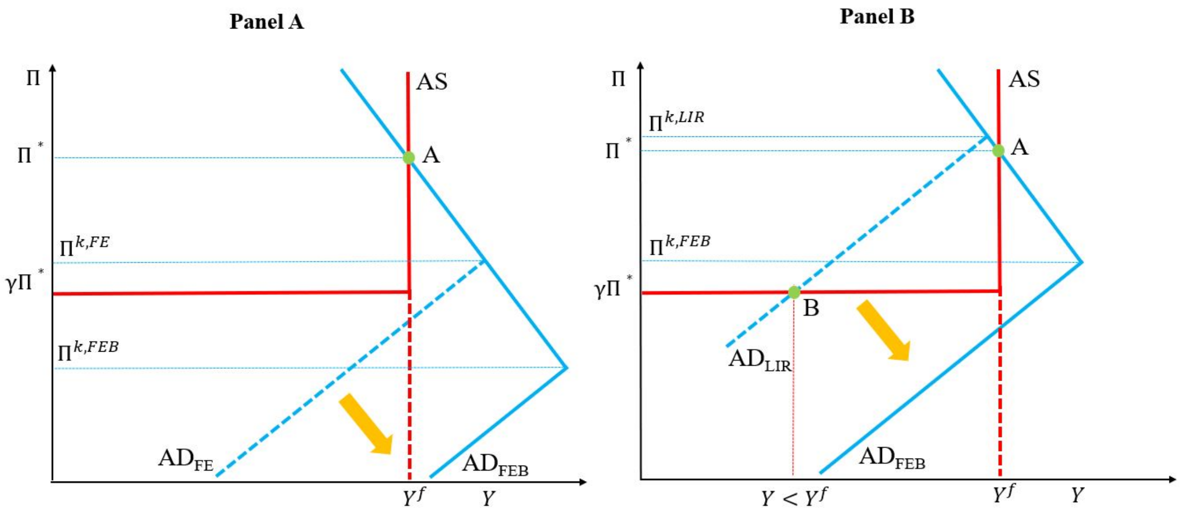
4.1.2 Bubbly Economy

As long as r_{nb} falls below g, a rational bubble can emerge in both the bubbleless equilibria considered. As bubbly assets have redistributive purposes, they do not alter the AS compared to the last section, but only the AD, whose new equations are given in Appendix B.3. More importantly, the equation of the AD kink is still given by (24), but the natural interest rate is higher, and Π^k is consequently lower in a bubbly economy than in a bubbleless one. This means the positive effect of asset price bubbles on the natural interest rate strongly mitigates the risk of hitting the ZLB for any given inflation level. Although the impact of bubbles on the ZLB constraint is overshadowed in normal times when the economy lies in a full employment equilibrium and the central bank can maneuver the policy rate freely, its importance emerges in an LIR environment when the binding ZLB causes a negative output gap due to demand shortage. This difference stands out in Figure 3, which depicts the shift from each of two bubbleless equilibria to the "full employment bubbly" (FEB) equilibrium. Specifically, in Panel A, the initial equilibrium is the FE one (point A), while it is the LIR equilibrium (point B) in Panel B.

If the bubble originates in the FE equilibrium, the new AD kink, $\Pi^{k,FEB}$, shifts the upward-sloping AD curve down from AD^{FE} to AD^{FEB}, but nothing changes substantially. FE and FEB equilibria occur at the same point, A. The emergence of bubbly assets cannot affect macroeconomic aggregates; rather, it raises the natural interest rate leading to higher real and nominal interest rates. The resulting extra space to maneuver the policy rate cannot be exploited by the central bank in

[12] Given $1 + r^f \geq 1$ and $\gamma\Pi^* \geq 1$, $\Pi^* \geq 1/\gamma(1 + r^f)$ and thus the FE equilibrium is unique, while the LIR one exists and it is unique because $1 + r^f < 1/\Pi^* < 1$ (Ascari and Bonchi, 2019). Furthermore, the "Taylor principle", $\phi_\pi > 1$, guarantees the determinacy of the FE equilibrium, while the LIR equilibrium is determinate because $\alpha * Y/(1 - \alpha) * (Y - D) > 0$. A formal proof for this condition is available upon request. As in Eggertsson et al. (2019), the LIR equilibrium does not suffer local indeterminacy, despite binding ZLB, because the presence of debt-constrained households prevents the emergence of an aggregate Euler equation similar to that of infinite-horizon models.

Figure 3: FEB equilibrium



Note: Color to be used.

normal times, but it becomes essential in an LIR equilibrium, as shown in Panel B of Figure 3. In such an environment, though redistributive bubbles cannot *per se* alter macroeconomic outcomes, they can drive the natural interest rate into non-negative/positive territory,

$$1 + r^f \geq 1, \tag{26}$$

and the central bank can consequently escape from the ZLB. In this case, the AD curve shifts from location AD_{LIR} to AD_{FEB}, and the economy moves from the original LIR equilibrium B to the FEB equilibrium A, in which $i > 0$, $Y = Y^f$ and $\Pi = \Pi^*$.[13]

As the natural interest rate depends on new and old bubbly assets in equation (20), there exists a minimum size of the bubble that makes equation (26) hold, and thus the economy reach the FEB equilibrium. I perform a standard calibration of the model, consistent with the recent estimates of the natural interest, to quantify this minimum size (Appendix B.4). Starting from the LIR equilibrium, a reasonably large aggregate bubble is sufficient for the economy to reach the new equilibrium, pointing to this result as general and not restricted to an extreme calibration of the bubble size. More

[13] The FEB equilibrium is unique under the same condition as the FE one: $\Pi^* \geq 1/\gamma(1 + r^f)$. Hence, given $\gamma\Pi^* \geq 1$, a strictly non-negative natural interest rate guarantees its existence and uniqueness. Only in the special case of $1 + r^f = \Pi^* = 1$, the central bank hits the ZLB in the FEB equilibrium, although it can stabilize inflation and output, $Y = Y^f$ and $\Pi = \Pi^*$. Finally, condition (26) guarantees the attainment of the new equilibrium also for a different DNWR that depends on the level of employment (Ascari and Bonchi, 2019, Appendix A.2).

precisely, if output increases immediately up to its potential level, the necessary initial (new) bubble size has to be less than 1% of GDP, and the aggregate bubble rises to 4.4% of GDP once the FEB equilibrium is reached. Furthermore, the associated increase in wealth, $(C^o_{FEB} - C^o_{LIR})/C^o_{LIR} = (U + B)/D$, is approximately 19% compared to the initial LIR equilibrium. To give an order of magnitude, the ratio of Wealth-to-GDP has been approximately twice as large (100% greater) as the Capital-to-GDP ratio in the US over the last decade (Piketty, Saez, and Zucman, 2018). I also consider the case in which the economy runs at its potential not immediately, but only when it reaches the FEB equilibrium. In this case, the minimum aggregate bubble size to achieve full employment is even lower in steady state (approximately 1.4% of GDP).

4.2 Welfare Analysis

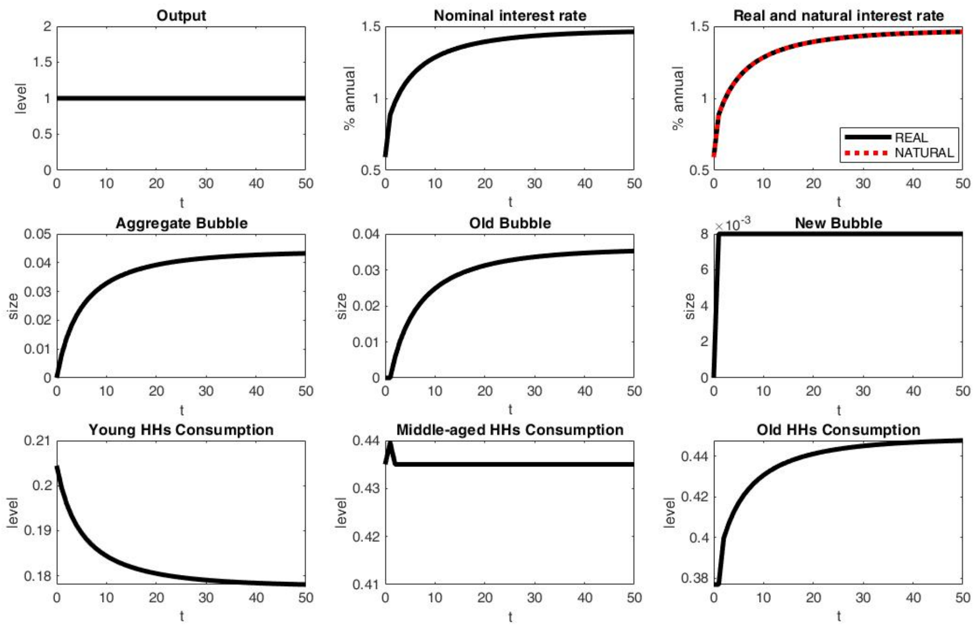
As output varies only when the economy moves from the LIR equilibrium to the FEB one, the welfare implications of bubbles could differ whether the ZLB is initially binding or not.

In Figure 4, I plot the transitional dynamics of the main variables from the FE equilibrium to the FEB one (Panel A) and from the LIR equilibrium to the FEB one (Panel B) to study the effect of bubbles on the intergenerational allocation of resources. The calibrated values for the parameters are those in Table 1 of Appendix B.4, and the only difference between FE and LIR equilibria is the calibration of g (0.025 and 0.016, respectively). At time zero, the economy is in the initial steady state equilibrium, and a (new) bubble emerges unexpectedly at time one. The constant size of the new bubble is $U = 0.008$, while the size of the old bubble at $t = 1$ is zero. Hence, the old bubble takes a positive value at $t = 2$, and its dynamic behavior is governed by (21). Finally, there is no innovation in the size of the old bubble, $B_t - E_{t-1}B_t = 0$ for all $t > 0$. Given these assumptions, agents have perfect foresight on the future evolution of the variables from time one onward.

Since the economy runs at the potential in the FE equilibrium, the emergence of asset bubbles does not affect output, but it raises interest rates, as depicted in the upper two rows of Panel A. Instead, starting from the LIR equilibrium, the increase in the natural rate engineered by the bubble leads the nominal interest rate into positive territory, with consequent output gains via higher aggregate demand (the upper two rows of Panel B). Specifically, for the chosen calibration of U, $r^f_t = 0$ at $t = 1$, and thus condition (26) holds immediately, with output "jumping" to the potential level.[14]

[14]Robustness exercises with alternative calibrations of U confirm the results (Appendix C.1.1). Furthermore, I assume constant zero inflation (not shown), $\Pi = \gamma\Pi^* = \Pi^* = 1$. Zero is a realistic lower bound on the targeted inflation ($\Pi^* \geq 1$) so that a non-negative (positive) natural interest rate delivers the potential output (and avoids the ZLB) regardless of the inflation target (condition 26). However, for a positive inflation target ($\Pi^* > 1$), the transition from the LIR to the FEB equilibrium can happen even if $r^f < 0$. The analysis of this case (Appendix C.1.2), which implies initial positive, below the

Panel A. From the FE to the FEB



Panel B. From the LIR to the FEB

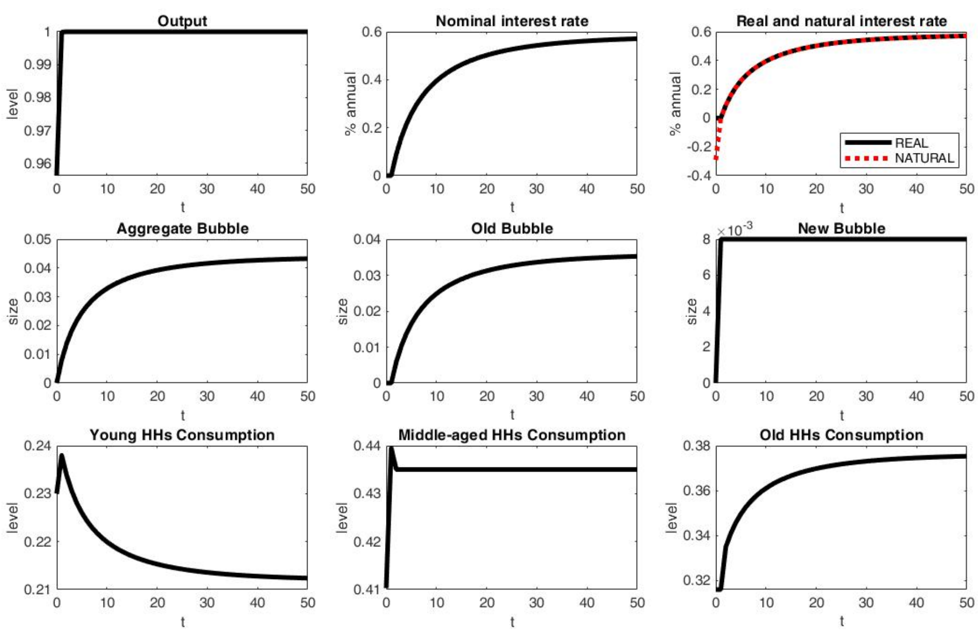


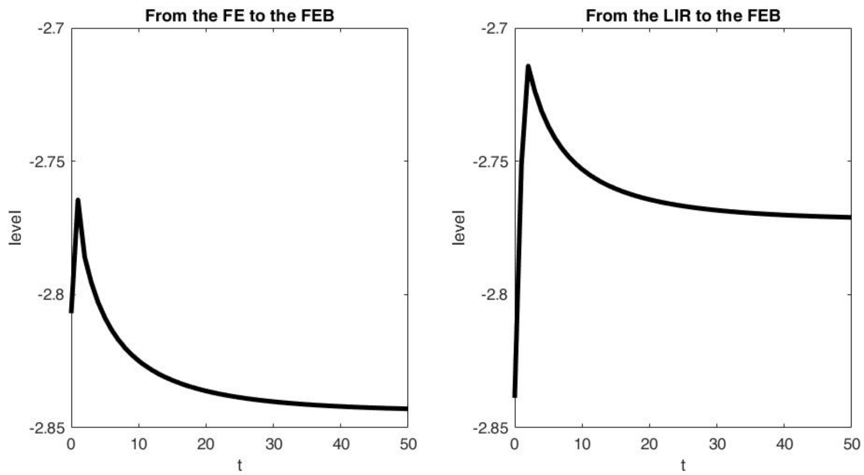
Figure 4: Transitional dynamics

The bottom row of the two panels shows that also the effect of bubbles on the intergenerational allocation of resources changes according to the initial bubbleless equilibrium. When the economy stays initially at the FE equilibrium, the occurrence of bubbles has only a redistributive effect (bottom row, Panel A). While the middle-age consumption is unchanged, except for a short-lived spike at $t = 1$, bubbles redistribute resources from the young generation to the old one. The central mechanism behind this intergenerational redistribution works through the amount and the allocation of savings, and the amount of debt, as explained above. In period one, the new bubble is a wealth shock that induces the middle generation to consume more by reducing savings/credit supply. Young households can pledge the (future) new bubble, but the credit supply is lower, and thus they raise fewer resources despite higher interest payments and debt to repay. Therefore, the young-age consumption declines, and it does consistently along the transition because the growing old/aggregate bubble absorbs an ever-larger share of the credit supply. On the other hand, the consumption of the old households raises from $t = 2$ onward. At $t = 1$, their consumption is unchanged because the old households do not carry over bubbles from period zero and earn interest payments from the loan contracts signed in that period. Instead, in the second period, the old generation receives a higher return from bonds, along with an additional income from the selling of the old bubble that emerged at $t = 1$. As the old bubble keeps growing until it converges to the steady state, the same happens to the consumption in old age.

Although the pattern of consumption across generations is similar if the economy originates in the LIR equilibrium, the magnitude and persistence of its variations are significantly different as a result of the output increase (bottom row, Panel B). More precisely, output influences the variations in young-age and middle-age consumption, while it does not influence the pattern of consumption in old age. Consumption in middle age still reaches a peak at $t = 1$, but now the increase is larger than that in Panel A due to a higher income. Furthermore, the increase is not short-lived. In the second period, middle-age consumption stabilizes at a level higher than the initial one because output rises permanently. Equally, young-age consumption no longer declines in the first period. Indeed, as the middle generation earns a higher income, it can raise both consumption and credit supply, with a consequent boost in the consumption of the young generation. However, as the old/aggregate bubble enlarges, it diverts an ever-larger share of saving from credit, reducing the increase in the credit supply. At some point along the transition (approximately five periods), the old bubble becomes so large that the credit supply, and thus young-age consumption, starts to decrease, and it keeps

target inflation, is substantially unchanged, except for the increase in inflation that occurs as output reaches the potential.

Figure 5: Welfare along the transition



declining until the bubble reaches the steady state. Overall, the *direct* redistributive effect of bubbles interacts with the *indirect* one on output, leading middle-aged and old households to consume more and young ones to consume less, apart from the initial periods of the transition.

As some generations are better off and others are worse off along the transition, I employ the lifetime utility of the representative agent as a welfare measure to establish the overall effect of bubbles. Specifically, I take the lifetime utility at time t of the middle-aged agent born at $t-1$,

$$\Omega_t = \ln C_{t-1}^y + \beta \ln C_t^m + \beta^2 \ln C_{t+1}^o, \tag{27}$$

because only the middle generation allocates consumption intertemporally, initiates the new bubbles and invests in the old ones. It is worth noting that the middle-aged agent at $t = 0$ is never affected by bubbles, also when it becomes old at $t = 1$, and thus its lifetime utility measures precisely that in the original bubbleless steady state. In Figure 5, I plot the pattern of welfare along the transition to the FEB starting from the two initial bubbleless equilibria.

In both panels, there is an initial peak in welfare associated with the emergence of the new bubble and the consequent peak in middle-age consumption. This initial welfare increase is higher, like that in middle-age consumption, if the economy is stuck initially at the ZLB, as in the right panel. From the second period onward, welfare starts to decline, following a similar pattern independently of the original bubbleless equilibrium. In the two cases, the consumption in middle age falls at $t = 2$ and the pattern of welfare becomes declining. Furthermore, the consumption in young age decreases, and that in old age increases from $t = 2$ up to the when transition is completed. Given the greater weight ($\beta < 1$) to the young-age consumption losses compared to the old-age consumption gains in the lifetime utility, welfare keeps declining for $t > 2$. Notwithstanding, the decline is more pronounced if the economy originates in the FE equilibrium, and the welfare level in steady state declines only in this case (left panel), while the welfare at the FEB equilibrium is higher than that at the LIR equilibrium (right panel).

The different response of output again causes these stark differences. Young-age consumption does not fall below the level at the LIR equilibrium immediately, but it is higher for some periods due to the higher credit supply caused by the output increase. Consequently, the young-age consumption losses are temporarily mitigated, implying a smoother declining pattern of welfare in the early periods.[15] On the other hand, in the second period, middle-age consumption reverts to the original level without any output increase, while it stabilizes permanently at a higher level when output rises. The different behavior of middle-age consumption explains why the welfare level in steady state is higher (lower) in the FEB equilibrium than that in the LIR (FE) one.[16]

To summarize, the model calibrations suggest that a bubble emerging in the FE equilibrium is welfare-reducing because it redistributes consumption from the young generation to the old one, and the welfare losses in young age are higher than the welfare gains in old age. By contrast, a bubble arising with LIR indirectly allows the economy to escape from the ZLB, with a consequent increase in output that raises the consumption of the middle generation, along with that of the old one, and thus overall welfare, despite the reduction in the consumption of the young generation.

5 Extensions

The previous analysis highlights the output and welfare gains from a redistributive asset price bubble at the ZLB. However, it studies the effect of bubbles on demand mediated by the natural interest rate, disregarding the interaction with other effects on the demand side and the supply side. Furthermore, it does not investigate the role of the bubble, whether the store of value or collateral, is crucial for

[15]In general, while the declining pattern of welfare derives from the different importance of consumption gains and losses in lifetime utility, the magnitude of the decline depends on the different marginal impact of consumption gains and losses due to the concavity of the utility function. As young-age consumption increases in the early periods, it starts to fall from a higher level, with a consequent muted marginal impact on utility.

[16]Starting from the LIR equilibrium, young-age consumption always increases initially. However, the persistence in the increase depends on U (Appendix C.1.1). For lower calibrations of U than the benchmark one, the aggregate bubble is smaller, and thus it diverts fewer savings away from credit, mitigating the declining pattern in young-age consumption after the initial peak or even delivering an increasing pattern. Hence, the benchmark calibration of U is conservative, and lower calibrations deliver a higher increase in welfare at the FEB equilibrium (the bottom panel of Figure 10).

producing output and welfare gains. This section first studies how the indirect effect of redistributive bubbles interacts with the "financial accelerator" and capital accumulation. Then, it investigates through which function the bubble mainly influences the natural interest rate.

5.1 Capital and Endogenous Fundamental Collateral

I add capital and endogenous fundamental collateral to the benchmark model. These new ingredients are described here, while referring to Appendix D.1 for a full explanation of the model. Specifically, I also introduce fiscal policy, and assume zero population growth to simplify the notation. The rest of the model is unchanged.[17]

On the firm side, capital is combined with labor through a Cobb-Douglas production function, and potential output is no longer fixed, but it depends on the capital stock accumulated in the previous period, $Y_t^f = K_{t-1}^{1-\alpha}\bar{L}^{\alpha}$. Production factors are remunerated at their marginal productivity due to perfect competition, and the "flexible" nominal wage is $P_t \alpha K_{t-1}^{1-\alpha}\bar{L}^{\alpha-1}$. Hence, the DNWR is given by:

$$W_t = \max\left(\gamma \Pi^* W_{t-1}, P_t \alpha K_{t-1}^{1-\alpha}\bar{L}^{\alpha-1}\right). \tag{28}$$

On the household side, the young generation accumulates capital, which fully depreciates after usage, to produce the next period. More precisely, young households borrow to invest in capital because its return net of taxes is higher than the real interest rate,[18] encouraging debt-financed capital investment as in Bacchetta et al. (2020). In the credit market, they can pledge bubbly collateral and a fundamental one, which is a fraction $\phi \in (0, 1-\tau)$ of the expected future income in middle age. Accordingly, the budget constraint at young age is

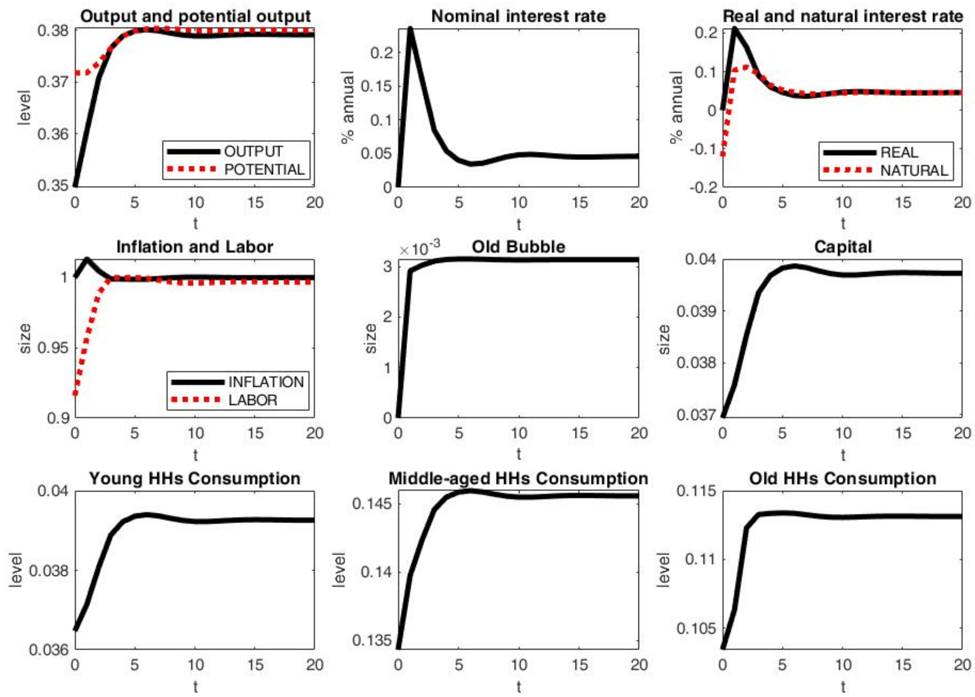
$$C_t^y + K_t = B_t^y = \frac{\phi E_t Y_{t+1} + \delta E_t P_{t+1|t+1}^B}{(1+r_t)}. \tag{29}$$

Since young households are borrowing constrained, their consumption and capital accumulation depend on the amount borrowed B_t^y, which, in turn, reflects the expected value of future income and new bubbles. The bubbly collateral crowds *in* capital and young-age consumption by fostering borrowing, but the amount of credit is limited by its supply, and bubble purchases crowd *out* capital and

[17] Although the introduction of fiscal policy makes the model more tractable analytically, I deliberately constrain its scope by assuming a constant public debt, $\bar{B}^g$, which is not sufficiently high to deliver a positive natural interest rate. The analysis of the interaction between fiscal and monetary policies and asset bubbles is left for future research.

[18] This is consistent with the pattern of the return on capital, which has been roughly stable in the last decades and thus disconnected from that of the risk-free real interest rate (Marx et al., 2019).

Figure 6: Transitional dynamics from the LIR to the FEB



consumption by diverting resources away from credit and discouraging saving. On the other hand, the endogenous fundamental collateral creates a "financial accelerator" mechanism. Any expected increase in output boosts the collaterals' value of young households, inducing them to consume and accumulate more capital, thus triggering a second-round effect working through aggregate demand (higher consumption) and supply (higher capital and thus potential output).[19]

Finally, there is an additional optimality condition equating the marginal cost of capital with the marginal benefit, given by its discounted expected return net of taxes and debt repayment:

$$\frac{1}{C_t^y}\left[1-\frac{\phi E_t r_{t+1}^k}{(1+r_t)}\right]=\beta E_t\left[\frac{(1-\tau-\phi)\, r_{t+1}^k}{C_{t+1}^m}\right], \tag{30}$$

where r_{t+1}^k is the return on capital and τ the tax rate on total (labor plus capital) income.

Once the extended model is illustrated, we can study the transition from the LIR equilibrium to

[19]Higher investment in capital does not positively affect aggregate demand because it reduces young-age consumption, for any given amount of borrowing in equation (29).

the FEB one (Figure 6). In addition to the calibrated parameters in Table 1 (Appendix B.4), I set $\phi = 0.21$ and $\tau = 0.09$. As in the previous section, the economy originates at the LIR equilibrium, and a constant new bubble ($U = 0.000034$) emerges at $t = 0$, while there is no initial old bubble and no innovation in its size along the transition. I assume a small new bubble because, if the economy nevertheless leaves the ZLB, amplification mechanisms are strong, which is precisely what I want to verify.[20]

The transitional dynamics of the main variables in Figure 6, though different from that in the benchmark specification (Panel B of Figure 4), does not lead to different conclusions. Rather, it highlights some new results that reinforce and clarify the previous ones.

On the one hand, the new bubble emergence triggers a "kick-off" effect by raising the natural interest rate and relaxing the ZLB. Though small initially, the effect of bubbles on demand will persist. Given perfect foresight, this causes a persistent appreciation of the fundamental collateral that fosters, along with the bubbly collateral, young-age consumption and investment in capital (equation (29)). In other terms, the fundamental collateral incorporates the cumulative increase in future demand, and thus "persistence and amplification reinforce each other" (Kiyotaki and Moore, 1997, p. 213). Furthermore, the growing aggregate bubble does not crowd out capital and young-age consumption because output/income in middle age rises greatly, allowing the middle generation to consume and save more. Hence, the larger amount of savings translates both into higher credit and bubble purchases. Overall, all generations, including the young one, are better off along the transition (bottom row of Figure 6), and the consumption gains are further amplified by higher capital and thus potential output in the final equilibrium (far-right panel in the middle row and far-left panel in the top row).[21]

On the other hand, the nominal interest rate does not monotonically increase in the exit from the ZLB. The strong increase in the generations' consumption boosts aggregate demand and output, taking the economy out of the ZLB immediately. However, the nominal interest rate spikes at impact, and then it declines to the new steady state level. Two different phenomena cause the initial spike.

[20]The calibration of ϕ falls in the range of values found in the literature (e.g., Iacoviello and Pavan, 2013; Guerrieri and Lorenzoni, 2017). The tax rate and public debt are calibrated to match, in the LIR equilibrium, the US general government final consumption in 2019 (14% of GDP, OECD). In the initial equilibrium, the chosen calibration delivers a 5.4% annual return on capital, which is very close to the return on equity estimated by Farhi and Gourio (2018) in 2001-2016 (4.9%). Finally, though the calibrated value for U is small, like the size of the economy (output is initially 0.35), the aggregate bubble in the final steady state is not very different from that in the previous section (1% vs 4% of GDP). In any case, I make some robustness exercises with higher values for U, which confirm my results and are available upon request.

[21]In this model, young households are similar to the "I-investors" in Bacchetta et al. (2020), while the middle-aged ones are a mix between "S-investors" (they earn the return from capital, repay debt and invest in assets) and "workers" (they earn labor income). Such a theoretical framework makes evident the central role of the bubbly collateral and the fundamental collateral depending on output (not on capital) in determining different results with respect to Bacchetta et al. (2020).

First, the natural interest rate overshoots its level in the FEB equilibrium, as shown by the equation:

$$1 + r_t^f = \frac{(1+\beta)\left(\phi Y_{t+1}^f + U_{t+1}\right)}{\beta\left[(1-\tau-\phi)\,Y_t^f - U_t - B_t\right] - U_t - B_t - (1+\beta)\,\bar{B}^g}, \tag{31}$$

where $U_{t+1} = U_t = U$ along the transition. While the term ϕY_{t+1}^f increases immediately due to the higher current capital accumulation, the term $(1-\tau-\phi)\,Y_t^f$ depending on past capital accumulation lags behind. Second, the initial spike is driven by the DNWR, which is still binding and determines the dynamics of inflation as follows:

$$\Pi_t = \gamma\Pi^* \left(\frac{L_t/L_{t-1}}{K_{t-1}/K_{t-2}}\right)^{1-\alpha}.$$

We can derive this equation from (28) by equating the (real) wage to its corresponding lower bound and replacing it with the marginal productivity of labor. At impact, the growth in employment/labor is more pronounced than that in capital (middle row, far-left and far-right panels), and thus inflation goes above its target level, $\Pi^* = \gamma\Pi^* = 1$. As a result of these two phenomena, the nominal interest rate reaches a level higher than its new steady state value. In particular, due to the inflation increase, this determines a real interest rate higher than its natural counterpart (top row, far-right panel). Hence, output does not immediately reach the potential level, which has risen due to capital accumulation (top row, far-left panel), but only after some periods when the inflation increase vanishes, and the policy rate is cut, consistently with the decline in the natural interest rate.

5.2 Unleveraged vs Leveraged Bubble

To study separately an *unleveraged* bubble, which serves only as a store of value, and a *leveraged* bubble, for which the role of bubbly collateral is prominent, I alter the structure of the financial markets in the benchmark model. Moreover, I abstract from population growth, $g = 0$, and normalize the size of generations to one, while firms and the monetary authority behave as in the benchmark framework.

In the market for bubbles, there exists one variety of bubbles in fixed unit supply. As there is an "old" bubble only due to the absence of bubble creation and destruction,[22] young households

[22]In this case, $\delta = 0$ so that new bubbles never emerge and old bubbles never lose value. Hence, the variety of bubbles is unique, and its supply is constant. It is worth noting that a unit supply of bubbles in aggregate is also the case in the benchmark model for $g = 0$ and $N_{t-1} = 1$ even if $\delta > 0$ ($Q_t^B = \sum_{j=0}^{\infty} Q_{t|t-j}^B = 1$).

have to participate directly in the market, along with the middle-aged households, if they want to borrow against the bubble. On the other hand, now the bubble seller can be both middle-aged and old households, which bought the bubbly asset in the previous period. Therefore, the equilibrium condition for the market of the bubble is

$$Q_t^{B,y} + Q_t^{B,m} = Q_{t-1}^{B,y} + Q_{t-1}^{B,m} = 1, \tag{32}$$

where $Q_t^{B,y}$ and $Q_t^{B,m}$ are respectively the bubble purchases of young and middle-aged households at time t. The price of the bubble, $\hat{P}_t^B$, is stochastic because each period there is a constant probability $\rho \in [0,1)$ the price collapses to 0 and, if the bubble has already collapsed, it cannot re-emerge again (Weil, 1987). Conditioning on not having crashed, the price of the bubble is $\hat{P}_t^B = P_t^B > 0$.

As regards the credit market, it is now incomplete because young households issue a one-period non-contingent debt contract that is defaultable. Furthermore, the real rate does not depend on the size of the loan, as in Allen and Gale (2000) and Ikeda and Phan (2016).[23] In the event of default, old households, who lent funds to young ones in middle age, can repossess only fundamental collateral, $D \in (0, Y)$, and bubbly collateral, which is a fraction ϕ of the young households' bubble holdings. The household's problem becomes:

$$\max_{C_{t+1}^m, C_{t+2}^o, Q_t^{B,y} \geq 0, Q_{t+1}^{B,m} \geq 0} E_t \left\{ \ln C_t^y + \beta \ln C_{t+1}^m + \beta^2 \ln C_{t+2}^o \right\}$$

s.t.

$$C_t^y = B_t^y - \hat{P}_t^B Q_t^{B,y}$$

$$C_{t+1}^m = Y_{t+1} + \hat{P}_{t+1}^B \left(Q_t^{B,y} - Q_{t+1}^{B,m} \right) - B_{t+1}^m - (1 - \xi_{t+1})(1 + r_t) B_t^y - \xi_{t+1} \left(D + \phi \hat{P}_{t+1}^B Q_t^{B,y} \right)$$

$$C_{t+2}^o = (1 - h_{t+2})(1 + r_{t+1}) B_{t+1}^m + \hat{P}_{t+2}^B Q_{t+1}^{B,m}$$

$$(1 + r_t) B_t^y = D + \phi P_{t+1}^B Q_t^{B,y}.$$

The notation is the same of Section 3, but now the representative household chooses the optimal bubble holding, which can be zero or positive, in young and middle age. Furthermore, middle-aged

[23] A microfoundation for the contract is given by Ikeda and Phan (2016). As the real rate charged on debt does not increase if young households borrow more via the bubbly collateral, risk-shifting is possible in equilibrium. In this way, the simplifying assumption of debt-inelastic interest rate replicates the effect of a "credit pool" (Bengui and Phan, 2018), in which loans with different default risk are pooled together, fostering risk-shifting from borrowers to lenders. As shown in Bengui and Phan (2018), even if the real return increases with debt, a leveraged bubble still emerges in the case of the bubble's full collateralization considered below.

households optimally choose to default or not at time $t+1$ according to the rule

$$\xi_{t+1} = \begin{cases} 0 & if\,(1+r_t)\,B_t^y \leq D + \phi \hat{P}_{t+1}^B Q_t^{B,y} \\ 1 & if\,(1+r_t)\,B_t^y > D + \phi \hat{P}_{t+1}^B Q_t^{B,y}. \end{cases}$$

The default function, ξ_{t+1}, is zero if middle-aged households do not go bankrupt because repaying the total amount of debt, $(1+r_t)\,B_t^y$, is at least as worthwhile as giving up fundamental and bubbly collaterals due to default. On the contrary, default occurs ($\xi_{t+1}=1$) if the outstanding debt is higher than the collaterals' value. As the total debt equals the maximum amount that can be repossessed by old households, $D + \phi P_{t+1}^B Q_t^{B,y}$, the middle generation opts for defaulting only when the bubble bursts.[24] Indeed, if this occurs, old households can only garnish D given that $\hat{P}_{t+1}^B = 0$. The remaining fraction of the original claims, $(1+r_t)\,B_t^m$, is a loss for old households, namely the haircut

$$h_{t+1} = \begin{cases} 0 & \xi_{t+1} = 0 \\ 1 - \frac{D}{(1+r_t)B_t^m} & \xi_{t+1} = 1 \quad (default). \end{cases} \tag{33}$$

5.2.1 Steady State Equilibrium

The steady state equilibrium is bubbleless for $P^B = 0$, while it is bubbly for $P^B > 0$. The bubbleless economy is identical to that in Section 4. Hence, leveraged and unleveraged bubbles could emerge in the LIR equilibrium and take the economy out of the ZLB through the natural interest rate. To study the two bubble types in isolation, I solve the household's problem in steady state by distinguishing a fully unleveraged bubbly equilibrium from a fully leveraged bubbly one, and thus by choosing a specific calibration of ϕ for each equilibrium. I report here the main results, referring the readers to Appendix D.2 for more details.

Unleveraged bubble. The bubble is fully unleveraged when the middle generation only owns it for investment purposes ($Q_t^{B,m} = 1$ and $Q_t^{B,y} = 0$). Therefore, the bubble is a store of value but not a collateral. For $\phi = 0$, borrowing against bubbly assets is impossible, and the bubble is fully unleveraged by construction.[25] In this case, the price of the unleveraged bubble, P^{BU}, and the

[24] As the old generation repossesses effectively the maximum amount only in the event of the bubble's survival, $\hat{P}_{t+1}^B = P_{t+1}^B > 0$, the borrowing constraint could be alternatively written as $(1+r_t)\,B_t^y = D + \phi max\{\hat{P}_{t+1}^B\}Q_t^{B,y}$. The constraint is binding either if the bubble cannot be or can be fully collateralized because $[1+\beta\,(1+\beta)]\,D < Y_t$ and

$$D < \frac{1+\beta}{1+\beta\,(1+\beta)\,(1-\rho)} \left\{ \frac{1}{1+\beta} Y_{t+1} + \beta\,(1-\rho) \left[P_t^B\,(1+r_t) - P_{t+1}^B \right] \right\}.$$

[25] Young households could demand bubbly assets too for $\phi = 0$ because these are complex securities that play for them the twofold role of collateral and store of value. However, considering $Q_t^{B,y} > 0$ would complicate the analysis without

corresponding real interest rate are, respectively,

$$P^{BU} = (1-\rho)\,\frac{\beta}{1+\beta}\,(Y-D) - D$$
$$(1+r) = \frac{(1+\beta)\,D}{\rho\beta\,(Y-D) + (1+\beta)\,D} < 1.$$

Looking at the first equation, the unleveraged bubbly equilibrium exists, $P^{BU} > 0$, only if $1-\rho > 1 + r_{nb} = (1+\beta)\,D/\beta\,(Y-D)$, a standard condition for rational stochastic bubbles (e.g., Bengui and Phan, 2018). As the real interest rate is negative in the initial LIR equilibrium, $r_{nb} < 0$, a fully unleveraged bubble can arise. More importantly, the unleveraged bubble does *not* make the natural interest rate, given by the second equation at $Y = Y^f$, non-negative/positive. Hence, the increase in the natural interest rate, engineered by the bubble, does not drive the economy out of the ZLB, as depicted in Panel B of Figure 3, but output and inflation gaps are still negative. However, the economy moves to a new LIR equilibrium, in which output increases slightly because a higher natural interest rate reduces the real interest rate gap.[26]

Leveraged bubble. The bubble is fully leveraged when the young generation buys all the bubbly assets to borrow against them ($Q_t^{B,m} = 0$ and $Q_t^{B,y} = 1$). For $\phi = 1$, this is the case because the high collateral value induces young households to demand the entire supply of bubbles, and the bubble is only a collateral.[27] The price of the leveraged bubble, P^{BL}, and the associated real rate are

$$P^{BL} = \frac{\beta}{1+\beta}\,(Y-D) - D$$
$$1 + r = 1 + r^f = 1.$$

A fully leveraged bubble emerges under a looser condition than a fully unleveraged one.This condition, for which $P^{BL} > 0$, is $1 > 1 + r_{nb} = (1+\beta)\,D/\beta\,(Y-D)$, and it is certainly satisfied in an LIR equilibrium featuring $r_{nb} < 0$. On the other hand, while an unleveraged bubble is not able to drive the natural interest rate into positive territory, the leveraged bubble pushes the originally negative natural interest rate up to zero and thus $1 + r^f = 1$. As the central bank can now stabilize

altering the fully unleveraged nature of the bubble.

[26]For $\gamma\Pi^* > 1$ and $\gamma < 1$, a special case emerges if $1 + r^f < 1$ and $1/(1+r^f) < \Pi^* < 1/\gamma(1+r^f)$ (Ascari and Bonchi, 2019). In this case, the unleveraged bubble makes an FEB equilibrium arise, along with a bubbly equilibrium with full employment and binding ZLB, but the LIR equilibrium survives in Panel B of Figure 3. As a consequence, three equilibria exist and a multiplicity problem arises. I report the equations for the AD in the unleveraged and leveraged bubbly equilibria in Appendix D.2.

[27]For $\phi = 1$, the price of the bubble, carried over to middle age, is fully exhausted by the higher debt to repay. Hence, the bubble can be used effectively as collateral only, though it is, in principle, also a store of value. A formal proof that the bubble is fully leveraged for $\phi = 1$ is available upon request.

output at the potential and inflation at the target, the economy escapes from the LIR equilibrium and moves to an FEB one.[28] Both results rely on the specific nature of the leveraged bubble.

As young households own the bubble, they can shift the risk of the bubble bursting by borrowing against it, and the *risk-shifting* distinguishes a leveraged bubble starkly from an unleveraged one. Middle-aged households invest their income in an unleveraged bubble only if the bubble survival is likely enough, and thus the risk of the bubble bursting is sufficiently low, $1-\rho > 1+r_{nb}$. Unlike middle-aged households, young households do not bear the cost of the bubble bursting and can invest even in extremely risky bubbles. Indeed, the probability of bursting ρ does not enter in the condition for the emergence of a leveraged bubble, $1 > 1+r_{nb}$. Moreover, the risk-shifting behavior induces the young generation to invest heavily in the leveraged bubble, whose price/size is accordingly larger than that of the unleveraged one ($P^{BL} > P^{BU}$), with a consequent more substantial upward pressure on the natural interest rate.

6 Conclusions

The persistence of low interest rates could significantly affect the implications of bubbles for the intergenerational allocation of resources and welfare.

Asset bubbles redistribute wealth across generations by increasing the net worth of the older cohorts, which benefit from large capital gains, and decreasing that of the younger cohorts, which borrow more via appreciated collaterals. In my benchmark model, for a positive nominal interest rate, this redistribution of wealth increases the consumption of the older cohorts at the expense of the younger ones, and the uneven distribution of gains/losses from asset bubbles away from the ZLB is welfare-reducing according to the model calibrations. Instead, the relationship that runs from asset bubbles to monetary policy highlights a unique role for bubbles when the ZLB binds, with radically different distributional implications. In this case, the redistributive effect of bubbles translates into a higher natural interest rate, resulting from the bubble serving as a store of value and collateral. As a result, the central bank has extra margin to maneuver the policy rate, output increases, and the

[28] These results confirm and reinforce the earlier intuitions of Bonchi (2019) in a two-period OLG model. In such a setting, he finds similar conditions for the existence of leveraged and unleveraged bubbles, but he shows that even the unleveraged bubble can bring the economy to an FE equilibrium, though only under stringent conditions. If $g > 0$, the leveraged bubble would make the natural interest rate strictly positive, namely $1+r^f = 1+g$, while we would obtain

$$1+r^f = (1+g)\frac{(1+\beta)D}{\rho\beta\left(Y^f - D\right) + (1+\beta)D}$$

for the unleveraged bubble. Hence, in a fully unleveraged bubbly equilibrium, the natural interest rate can still be negative, $1+r^f < 1$, if the probability of bubble bursting ρ is high and/or the economy's growth rate is low.

gains from bubbles spread more uniformly across the age cohorts because of the higher income for workers in middle age, with consequent welfare gains.

In any case, the peculiar role of asset price bubbles at the ZLB parallels, though in the opposite direction, the conventional one as a source of macroeconomic instability. Therefore, an exhaustive analysis of bubbly episodes is required to account for the potential output losses in the bust phase. In this paper, I take the first step in this direction by investigating what role of bubbles influences mainly the natural interest rate. My extended OLG model suggests that the output gains from bubbly episodes, at the ZLB, go hand-in-hand with the output losses, because the bubble type that most increases output, the leveraged one, is also the most detrimental for it in case of bursting (Jordá et al., 2015). However, the model is silent about the magnitude of output gains/losses from different bubbly episodes and the weights attached to them from monetary and macroprudential authorities.

To address these issues, the analysis presented here should be extended in at least two dimensions. First, temporary bubbly episodes should be explicitly studied to evaluate correctly the output gains/losses from the bursting of leveraged and unleveraged bubbles. Second, and related to the first point, the trade-off from bubbly episodes at the ZLB should be studied within a richer model such as a "perpetual youth" OLG model (e.g., Galí, 2021). Indeed, in a standard OLG model, the interpretation of a period as the length of a generation, namely twenty or thirty years, is not appropriate for analyzing bubbly episodes that last just a few quarters.

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Appendix

A The Benchmark Model

A.1 Exogenous Productivity Growth

I spell out a different version of the benchmark model. The population is constant and the size of generations is normalized to one, but the economy still grows at a positive rate of g because of the exogenous growth in productivity ($A_{t+1}/A_t = 1 + g$). I denote the growth rate of productivity as that of the population in the main text to emphasize the equivalence between the two models. The production technology of firms is $Y_t = A_t L_t^{\alpha}$, while the labor demand is $\frac{W_t}{P_t} = \alpha A_t L_t^{\alpha-1}$.

The assumptions about the household sector are the same as Section 3, except for the quantity of bubbly assets that does not grow anymore and the exogenous debt limit that becomes $A_{t+1}D$. As productivity growth affects the income positively in middle age, enhancing the ability to repay debt, young households can now raise more funds in the credit market. As a consequence, the maximization problem of the representative household is identical to that in Section 3.1, apart from the two budget constraints:[29]

$$C_t^y = B_t^y = \frac{A_{t+1}D + \delta E_t P_{t+1|t+1}^B}{(1 + r_t)}$$

$$C_{t+2}^o = (1 + r_{t+1}) B_{t+1}^m + (1 - \delta) \sum_{j=0}^{\infty} P_{t+2|t+1-j}^B Q_{t+1|t+1-j}^B.$$

The new budget constraints do not alter the Euler equation (5), but the price of a generic bubbly asset becomes

$$P_{t|t-j}^B = (1 - \delta) \beta E_t \left[\left(\frac{C_t^m}{C_{t+1}^o} \right) P_{t+1|t-j}^B \right]. \tag{34}$$

I keep all the other assumptions of the benchmark model, as, for example, the DNWR (10) and the monetary policy rule (11). Turning to the two markets for financial assets, the equilibrium condition in the market for bubbles is (13), but now the aggregate supply of bubbles is one: $Q_t^B = \delta \sum_{j=0}^{\infty} (1 - \delta)^j = 1$. Notwithstanding, the bubble indexes normalized by productivity,

$$\tilde{P}_t^B = \tilde{B}_t + \tilde{U}_t = \beta (1 + g) \left[\left(\frac{C_t^m}{C_{t+1}^o} \right) \tilde{B}_{t+1} \right]$$

[29]Moreover, the borrowing constraint binds because $A_t D < [Y_t - \beta (1 + \beta) \delta P_{t|t}^B]/[1 + \beta (1 + \beta)]$.

$$\tilde{B}_t = \frac{1}{A_t}\delta \sum_{j=1}^{\infty} (1-\delta)^j P^B_{t|t-j}$$

$$\tilde{U}_t = \frac{\delta P^B_{t|t}}{A_t},$$

are equivalent to those normalized in terms of N_{t-1} of Section 3.4. The symbol ˜ denotes variables normalized by productivity, and all the expectation terms are suppressed due to perfect foresight.

A.2 Market for Equity

I extend the model of Appendix A.1 by assuming middle-aged households can also invest in the shares of firms, which are still finitely-lived but can now be active for more than one period. It would be equivalent to extend the model of Section 3, as the exogenous source of growth does not alter the structure of the benchmark model. However, the equity market complicates the notation greatly, and the analysis remains clear and simple by working with the model of Appendix A.1, which features no growth in the bubbles quantity and fixed unit supply of bubbles in aggregate.

In each period, middle-aged households set up new firms, whose shares are δ, and δ old firms, which were active from the earlier periods, shut down. The middle generation is accordingly endowed with the new firms' shares, while it buys the old firms' shares from the old generation. The profits of all firms are distributed to the middle generation investing in their shares.[30] The household's maximization problem is:[31]

$$\max_{C^m_{t+1}, C^o_{t+2}, Q^F_{t+1|t+1-j}, Q^B_{t+1|t+1-j}} E_t \left\{ \ln C^y_t + \beta \ln C^m_{t+1} + \beta^2 \ln C^o_{t+2} \right\}$$

s.t.

$$C^y_t = B^y_t = \frac{A_{t+1}D + \delta E_t \left(P^F_{t+1|t+1} + P^B_{t+1|t+1} \right)}{(1+r_t)}$$

$$C^m_{t+1} = \frac{W_{t+1}}{P_{t+1}} L_{t+1} + \delta \left(P^F_{t+1|t+1} + P^B_{t+1|t+1} \right) - (1+r_t) B^y_t - B^m_{t+1} - \sum_{j=0}^{\infty} \left(P^F_{t+1|t+1-j} - Z_{t+1} \right) Q^F_{t+1|t+1-j} - \sum_{j=0}^{\infty} P^B_{t+1|t+1-j} Q^B_{t+1|t+1-j}$$

[30] This assumption about the distribution of profits, along with that of finitely-lived firms, keeps the aggregate value of shares finite for a real interest rate lower than the economy's growth rate, allowing the simultaneous existence of markets for shares and bubbles as in Galí (2021).

[31] In this case, the borrowing constraint binds for $A_t D < \left[\frac{W_t}{P_t} L_t - \beta (1+\beta) (\delta P^F_{t|t} + \delta P^B_{t|t}) \right] / [1 + \beta (1+\beta)]$.

$$C_{t+2}^{o} = (1 + r_{t+1})\, B_{t+1}^{m} + (1 - \delta) \left[\sum_{j=0}^{\infty} P_{t+2|t+1-j}^{F} Q_{t+1|t+1-j}^{F} + \sum_{j=0}^{\infty} P_{t+2|t+1-j}^{B} Q_{t+1|t+1-j}^{B} \right].$$

$P_{t|t}^{F}$ is the price of the new firms' shares. Instead, $P_{t|t-j}^{F}$ and $Q_{t|t-j}^{F}$ are the price and quantity, at time t, of the shares for the firm set up at time $t-j$. As the middle generation owns the new firms' shares, young households can also pledge the expected value of future new firms to raise resources in the credit market. In addition to equations (5) and (34), we have an optimality condition for the price of shares,

$$P_{t|t-j}^{F} = Z_t + (1 - \delta)\, E_t \left[\beta \left(\frac{C_t^{m}}{C_{t+1}^{o}} \right) P_{t+1|t-j}^{F} \right], \tag{35}$$

which depends on profits, and not only on its expected future value, because shares are not worthless.

We turn to the market for assets. The market for the shares of the firm set up at time $t-j$ clears if the demand from the middle generation equals the supply from the old one:

$$Q_{t|t-j}^{F} = \delta\, (1 - \delta)^{j}. \tag{36}$$

Given the exit process of firms, the old firms' shares decline over time, but the aggregate supply of shares is constant and equal to one, $Q_t^F = \sum_{j=0}^{\infty} Q_{t|t-j}^{F} = 1$, because in each period the new firms' shares offset precisely those of the exiting firms. We can define the aggregate stock index normalized by productivity:

$$\tilde{P}_t^{F} = \frac{\delta \sum_{j=0}^{\infty} (1 - \delta)^{j} P_{t|t-j}^{F}}{A_t} = \tilde{P}_t^{F,N} + \tilde{P}_t^{F,I}. \tag{37}$$

The market capitalization of all firms is given by the sum of the market capitalization of old firms (the superscript I stands for "incumbents"),

$$\tilde{P}_t^{F,I} = \frac{\delta \sum_{j=1}^{\infty} (1 - \delta)^{j} P_{t|t-j}^{F}}{A_t}, \tag{38}$$

and that of new firms,

$$\tilde{P}_t^{F,N} = \frac{\delta P_{t|t}^{F}}{A_t}. \tag{39}$$

Alternatively, given perfect foresight, the aggregate stock index can be expressed as

$$\tilde{P}_t^{F} = \tilde{Z}_t + (1 + g) \left[\frac{\tilde{P}_{t+1}^{F,I}}{(1 + r_t)} \right] \tag{40}$$

through the equations (5), (35), (37) and (38). The market for bubbles is unchanged compared to Appendix A.1. As regards the market for credit, the demand and supply normalized by productivity are respectively

$$\tilde{D}_t^c = \tilde{B}_t^y = (1+g)\left[\frac{D + \tilde{P}_{t+1}^{F,N} + \tilde{U}_{t+1}}{(1+r_t)}\right]$$

and

$$\tilde{S}_t^c = \tilde{B}_t^m = \frac{\beta}{1+\beta}\left(\tilde{Y}_t - D - \tilde{P}_t^F - \tilde{U}_t - \tilde{B}_t\right) - \frac{1}{1+\beta}\left(\tilde{P}_t^F + \tilde{U}_t + \tilde{B}_t - \tilde{Z}_t\right),$$

where the two equations are obtained by following the same steps of Section 3.[32] Finally, the equilibrium real interest rate is given by

$$(1+r_t) = \frac{(1+g)(1+\beta)\left(D + \tilde{P}_{t+1}^{F,N} + \tilde{U}_{t+1}\right)}{\beta\left(\tilde{Y}_t - D - \tilde{P}_t^F - \tilde{U}_t - \tilde{B}_t\right) - \left(\tilde{P}_t^F + \tilde{U}_t + \tilde{B}_t - \tilde{Z}_t\right)}. \tag{41}$$

Apart from the additional conditions implied by the market for equity (and productivity growth instead of population growth to simplify the notation), the model and the related results are the same as in Section 3. The introduction of equity shares, as an alternative store of value and additional collateral, reduces the credit supply and increases the credit demand, but the real interest rate can still be negative. Moreover, the bubble exerts upward pressure on the real interest rate through the same channels outlined in the main text. The interpretation of the bubble only changes in this setting because, though it is still a specific asset, we can now interpret it as the component of the shares' price unrelated to the fundamentals. Specifically, the bubble creation can be associated with the setting up of new firms, while its destruction with the exiting of the old firms from the goods market. This interpretation helps to clarify why nothing changes, compared to the model of Section 3.

B Steady State Equilibrium

B.1 Existence of the Bubbly Equilibrium

This proof is very close to that in Galí (2014, Appendix 2), so I refer to that paper for further details. H mapping has the following properties.

[32]For the computation, we use $(1-\delta)\sum_{j=0}^{\infty} P_{t+1|t-j}^F Q_{t|t-j}^F / A_{t+1} = \tilde{P}_{t+1}^{F,I}$, $\sum_{j=0}^{\infty} P_{t|t-j}^F Q_{t|t-j}^F / A_t = \tilde{P}_t^{F,N} + \tilde{P}_t^{F,I}$, $\sum_{j=0}^{\infty} Q_{t|t-j}^F = 1$, as well as equations (39) and (40).

1. $H(B,U) \geq 0$ is twice continuously differentiable for $0 \leq B < \bar{B}(U)$, where $\bar{B}(U) = \frac{\beta}{1+\beta}(Y-D) - U$. If $B > \bar{B}(U)$, $H(B,U) < 0$.

2. The derivatives of $H(B,U)$ with respect to B_t are

$$\frac{\partial H(B,U)}{\partial B_t} = \frac{\beta(1+\beta)(D+U)(Y-D)}{[\beta(Y-D-U-B)-(U+B)]^2} > 0$$

and

$$\frac{\partial^2 H(B,U)}{\partial B_t^2} = \frac{2\beta(1+\beta)^2(D+U)(Y-D)}{[\beta(Y-D-U-B)-(U+B)]^3} > 0.$$

The second inequality holds for $0 \leq B < \bar{B}(U)$ and $\lim_{B\to\bar{B}(U)} H(B,U) = +\infty$.

3. The derivatives of $H(B,U)$ with respect to U are

$$\begin{aligned}\frac{\partial H(B,U)}{\partial U} =& \frac{(1+\beta)(D+2U+B)[\beta(Y-D-U-B)-(U+B)]}{[\beta(Y-D-U-B)-(U+B)]^2} + \\ & \frac{(1+\beta)^2(D+U)(U+B)}{[\beta(Y-D-U-B)-(U+B)]^2} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 H(B,U)}{\partial U^2} =& \frac{2(1+\beta)[\beta(Y-D-U-B)-(U+B)]^2}{[\beta(Y-D-U-B)-(U+B)]^3} + \\ & \frac{2(1+\beta)^2(D+2U+B)[\beta(Y-D-U-B)-(U+B)]}{[\beta(Y-D-U-B)-(U+B)]^3} + \\ & \frac{2(1+\beta)^3(D+U)(U+B)}{[\beta(Y-D-U-B)-(U+B)]^3} > 0.\end{aligned}$$

Both inequalities hold for $0 \leq B < \bar{B}(U)$ and $\lim_{B\to\bar{B}(U)} H(B,U) = +\infty$.

4. The mixed second derivative is

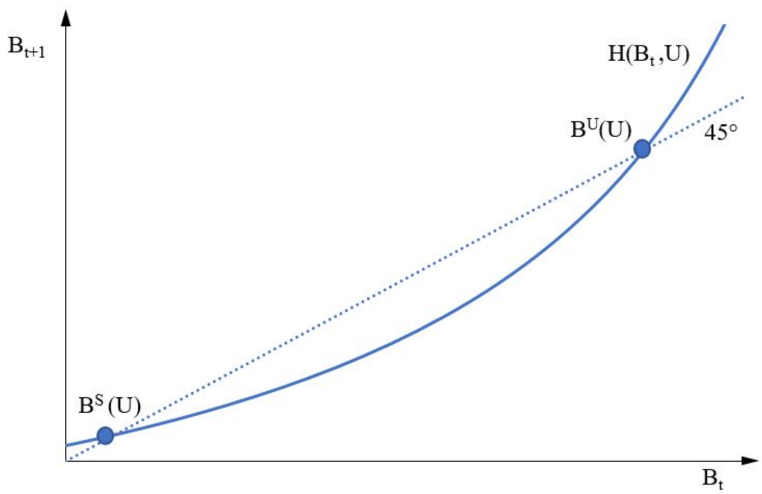
$$\frac{\partial H(B,U)}{\partial B_t \partial U} = \frac{\beta(1+\beta)(Y-D)\{[\beta(Y-D-U-B)-(U+B)] + 2(1+\beta)(D+U)\}}{[\beta(Y-D-U-B)-(U+B)]^3},$$

and it is positive for $0 \leq B < \bar{B}(U)$ and $\lim_{B\to\bar{B}(U)} H(B,U) = +\infty$.

Consider first the case $U = 0$. Equation (21) becomes

$$B_{t+1} = \frac{(1+\beta)DB_t}{\beta(Y-D-B_t)-B_t} = H(B_t, 0).$$

Figure 7: Old bubble dynamics



A solution to this equation is the *bubbleless* steady state equilibrium $(B, U) = (0, 0)$. A *bubbly* steady state equilibrium $(B^U, 0)$ with $B^U \in (0, Y^f)$ is another solution of the equation if

$$\frac{\partial H(0,0)}{\partial B_t} = \frac{(1+\beta) D}{\beta (Y - D)} < 1.$$

This condition, which is necessary and sufficient for the existence of the bubbly equilibrium $(B^U, 0)$, derives from property 2, and it can be alternatively expressed as

$$D < \frac{\beta}{1+\beta} (Y - D).$$

The bubbly equilibrium is unstable for the same reasons expressed in Galí (2014).

Sufficiency: Assume condition (22) holds. Given property 3 and the continuity of H, there are two steady state equilibria $B^U(U)$ and $B^S(U)$ for any $U \in (0, \bar{U})$ with $\bar{U} = [(1+2\beta) D - \beta Y]^2 / [4\beta (1+\beta)(Y - D)]$. $B^U(U)$ and $B^S(U)$ have the same stability properties of the equilibria in Galí (2014), and $B^U(U) > B^S(U)$. These two equilibria are depicted in Figure 7.

Necessity: The proof is equivalent to that in Galí (2014).

B.2 Stationarity of the Old Bubble

Taken at t, equation (21) becomes

$$B_t = \frac{(1+\beta)(D+U_t)(U_{t-1}+B_{t-1})}{\beta(Y-D-U_{t-1}-B_{t-1})-(U_{t-1}+B_{t-1})}.$$

Denoting log-linearized variables by lowercase letters, the log-linearized version of the equation above is

$$b_t = \varphi v_b b_{t-1} + \psi_u u_t + \varphi v_u u_{t-1},$$

where $\varphi = \frac{\beta(Y-D)}{[\beta(Y-D-U-B)-(U+B)]}$, $v_b = \frac{B}{U+B}$, $\psi_u = \frac{U}{D+U}$ and $v_u = \frac{U}{U+B}$. The condition for the stationarity of the old bubble is

$$\varphi v_b = \left[\frac{\beta(Y-D)}{\beta(Y-D-U-B)-(U+B)}\right]\left(\frac{B}{U+B}\right) < 1,$$

and it coincides with that for the stability of the bubbly equilibrium

$$\frac{\partial H(B,U)}{\partial B} = \left[\frac{\beta(Y-D)}{\beta(Y-D-U-B)-(U+B)}\right]\left[\frac{(1+\beta)(D+U)}{\beta(Y-D-U-B)-(U+B)}\right] < 1.$$

Indeed,

$$\left(\frac{B}{U+B}\right) = \left[\frac{(1+\beta)(D+U)}{\beta(Y-D-U-B)-(U+B)}\right]$$

follows directly from equation (21) taken at the bubbly steady state.

B.3 Asset Price Bubbles and the ZLB: Aggregate Demand

B.3.1 Bubbleless Economy

In a bubbleless economy, combining equations (11), (12), and (20) yields the following AD with a positive policy rate ($1+i>1$):

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta}\right)\left(\frac{1+g}{1+r^f}\right)\left(\frac{\Pi^*}{\Pi}\right)^{\phi_\pi - 1} D. \tag{42}$$

If $1+i=1$, we get a different AD from the equations above:

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta}\right)(1+g)\,\Pi D. \tag{43}$$

B.3.2 Bubbly Economy

In a bubbly economy, the AD becomes

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta}\right)(U+B) + \left(\frac{1+\beta}{\beta}\right)\left(\frac{1+g}{1+r^f}\right)\left(\frac{\Pi^*}{\Pi}\right)^{\phi_\pi - 1}(D+U), \tag{44}$$

for a positive policy rate, and it is

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta}\right)(U+B) + \left(\frac{1+\beta}{\beta}\right)(1+g)\,\Pi\,(D+U) \tag{45}$$

in case of binding ZLB.

B.4 Calibrated Model

I calibrate the benchmark model to study the transitional dynamics from the LIR equilibrium to the FEB one. The economy stays at the original steady state at time zero. The new bubble, which is constant over time, emerges at $t = 1$ ($U_t = U > 0$ for all $t > 0$), while the size of the old bubble, which evolves according to (21), is initially zero ($B_1 = 0$ and $B_t > 0$ for all $t > 1$). There is no innovation in the size of the old bubble, $B_t - E_{t-1}B_t = 0$ for all $t > 0$.

The upper part of Table (1) contains the calibrated values for the parameters. These values are in annual terms, and they have to be converted to 20 years, which is the assumed length of a generation. A similar consideration applies to the other calibrations of the model described in the paper. Labor supply is one to normalize all variables in terms of potential output ($Y^f = \bar{L}^\alpha = 1$), which coincides with current output in the FEB equilibrium. The economy's growth rate, $g = 0.016$, is approximately the average real GDP growth rate in 2000-2020, computed by the IMF (2020) for the advanced economies (0.013 or 1.3%). $\gamma = 1$ falls in the range of values found by Schmitt-Grohé and Uribe (2016), while I set D to the value of the quantitative model of Eggertsson et al. (2019). The remaining parameters are standard.

In a bubbleless economy, this calibration delivers $r^f_{nb} = -0.003$ (-0.3%), which is very close to the average US natural interest rate over the period 2000-2019, -0.67%, estimated at a quarterly frequency by Cúrdia (2015). The estimates of Cúrdia (2015) are a useful benchmark, because the theoretical model used for their computation incorporates a short-term definition of the natural interest rate that fits with that in my model. For the assumed zero inflation target, $\Pi^* = 1$, the level of natural interest rate obtained is compatible with an LIR equilibrium (condition (25) holds), which

Table 1: Parameters and variables

Parameters	**Values**	**Description**
β	0.987	Discount factor
α	0.7	Labor share
D	0.23	Collateral constraint
g	0.016	Growth rate
Π^*	1	Inflation target
ϕ_π	2	Taylor coefficient
γ	1	Wage rigidity
$\bar{L}$	1	Labor supply

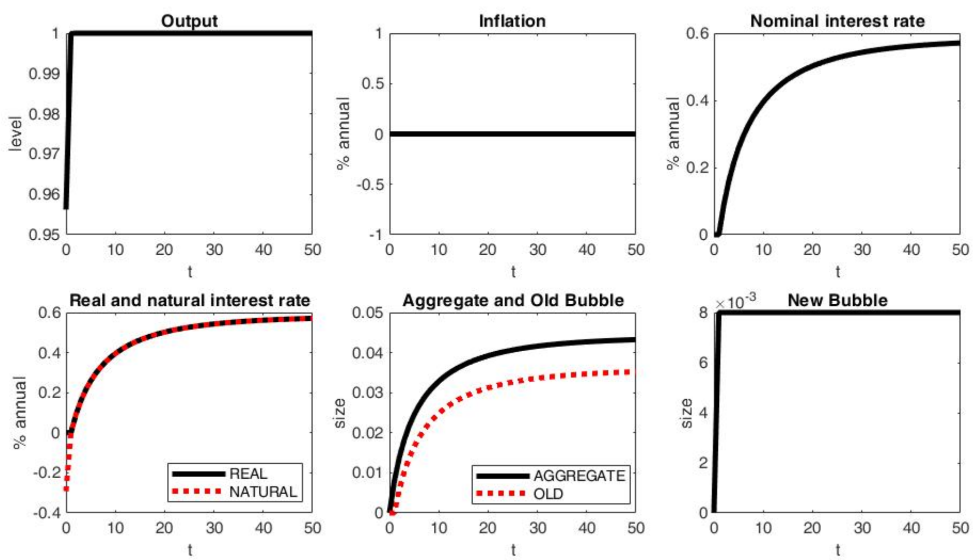
Variables	**LIR**	**FEB 1**	**FEB 2**
Y	0.956	1	1
Π	1	1	1
i	0	0.0058	0
r	0	0.0058	0
r^f	-0.003	0.0058	0
U	0	0.008	0.00375
B	0	0.036	0.01
$P^B = B + U$	0	0.044	0.01375

features $\Pi = \gamma\Pi^* = \Pi^* = 1$.

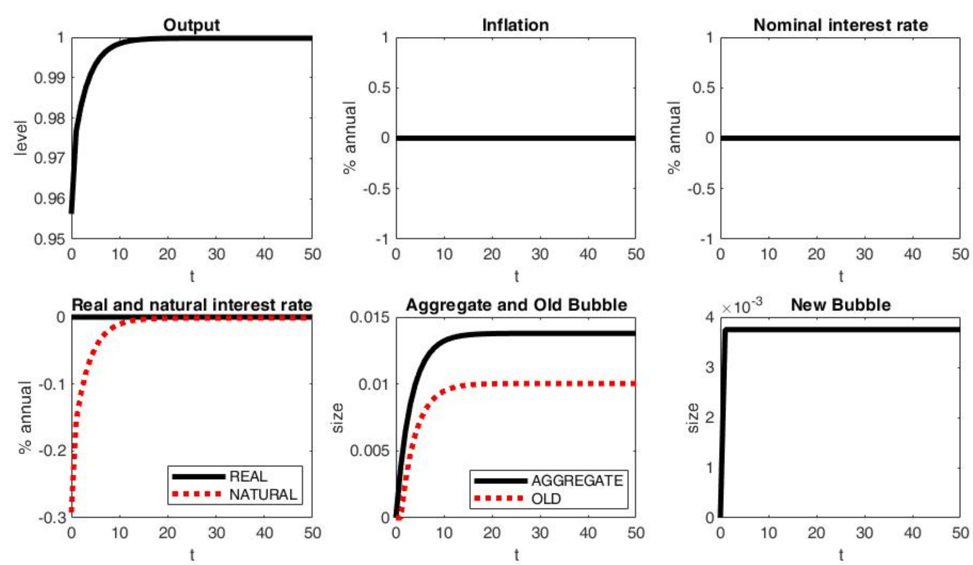
Starting from the LIR equilibrium, the economy reaches the FEB one once the natural interest rate turns non-negative/positive, $1 + r^f \geq 1$. A zero natural rate of interest is sufficient to move the economy to the FEB equilibrium because the central bank can equate the real interest rate to its natural level and output to its potential level, though the nominal interest rate remains stuck at the ZLB. More precisely, output increases immediately up to the potential, if the emergence of the new bubble at time one makes the natural interest rate zero, and thus the size of the initial aggregate bubble, $P_1^B = U_1 = U$, is sufficiently large to deliver $r_1^f = 0$. Instead, the economy runs at the potential level only after the transition to the FEB equilibrium is completed, if the aggregate bubble in steady state, $P^B = U + B$, is just sufficient to deliver $r^f = 0$. The aggregate bubble converges to the steady state when the old bubble does. These two cases are of interest because the first one corresponds to the minimum *initial* aggregate bubble that takes output immediately to the potential without the transition being completed, and the second one is associated with the minimum size of the *steady state* aggregate bubble that delivers the potential output after the full transition.

Figure 8: Transitional dynamics from the LIR to the FEB

(a) **FEB 1**.



(b) **FEB 2**.



I refer to the first case as "FEB 1" and to the second one as "FEB 2", and I plot the corresponding transitional dynamics, respectively, in the top and bottom panel of Figure 8. As already mentioned, output rises to its potential as the new bubble emerges in the case of FEB 1. Although the new bubble is constant, the aggregate bubble keeps growing until the old bubble reaches its steady state value. Therefore, the natural and real interest rates increase well above zero, taking the economy out of the ZLB. By contrast, the economy converges to FEB 2 after approximately 15 periods, when the old and aggregate bubbles stabilize to their steady state values corresponding to a zero natural interest rate, and thus output reaches its potential level. Along the transition, the natural interest rate increases due to the growing old bubble, reducing the gap with the real interest rate and thus stimulating output.

In the bottom part of Table 1, I report the steady state values of the main variables for FEB 1, FEB 2 and the initial LIR equilibrium. The minimum size of the new bubble to achieve the potential output immediately is small (0.008 or 0.8% of GDP), likewise the aggregate bubble in the FEB 1 (0.044 or 4.4% of GDP). A similar consideration applies to the minimum size of the aggregate bubble to reach the potential output in steady state (approximately 0.014 or 1.4% of GDP for the FEB 2). Furthermore, the increase in wealth from the LIR equilibrium to FEB 1 and FEB 2, $(C^o_{FEB} - C^o_{LIR})/C^o_{LIR} = (U + B)/D$, is low (respectively, 19.13% and 6%).

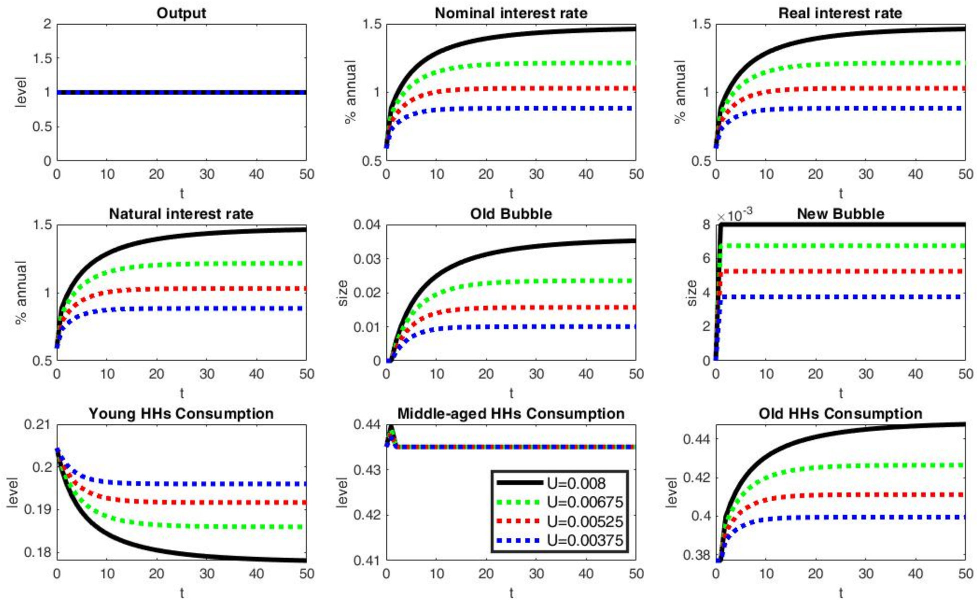
C Welfare Analysis

C.1 Alternative Calibrations

C.1.1 New Bubble Size

I repeat the analysis of Section 4.2 for alternative values of the new bubble. As the benchmark value for U, 0.008, is very close to the upper bound $\bar{U}$ (see Appendix B.1), the alternative calibrations imply a smaller new bubble, which is, in any case, large enough to bring the economy to the FEB equilibrium starting from the LIR one (the lowest calibration corresponds to "FEB 2" in Appendix B.4). I plot the transition(s) from the FE (Panel A) and the LIR equilibrium (Panel B) to the FEB one in Figure 9, where the thick black line denotes the benchmark calibration. I do not show the pattern of the aggregate bubble that closely follows that of the old one.

Although there are some changes in the variables' pattern for different U, the overall picture is substantially unchanged with respect to the benchmark calibration. If the new bubble shrinks, Panel A. From the FE to the FEB



Panel B. From the LIR to the FEB

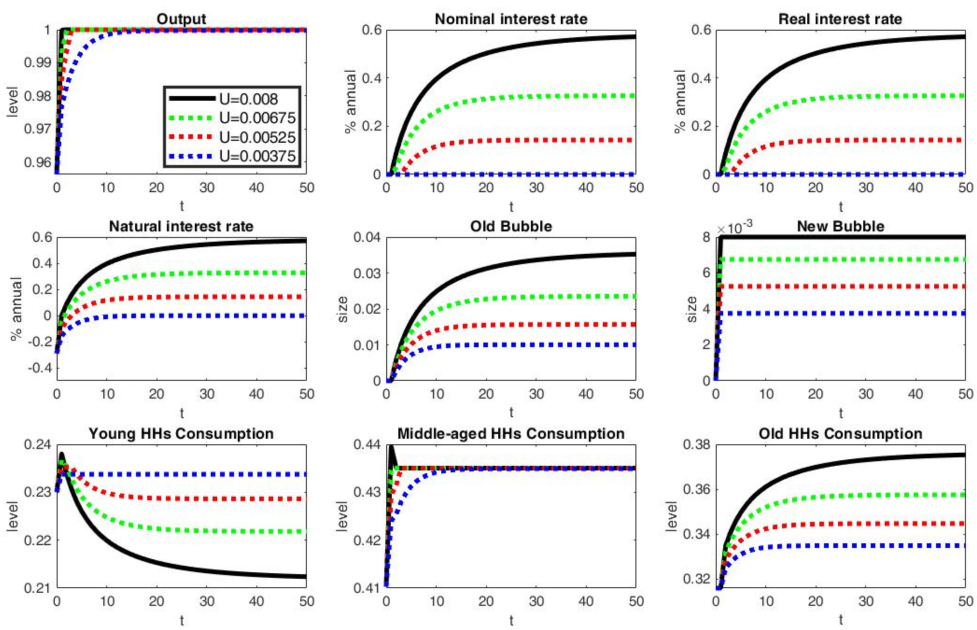
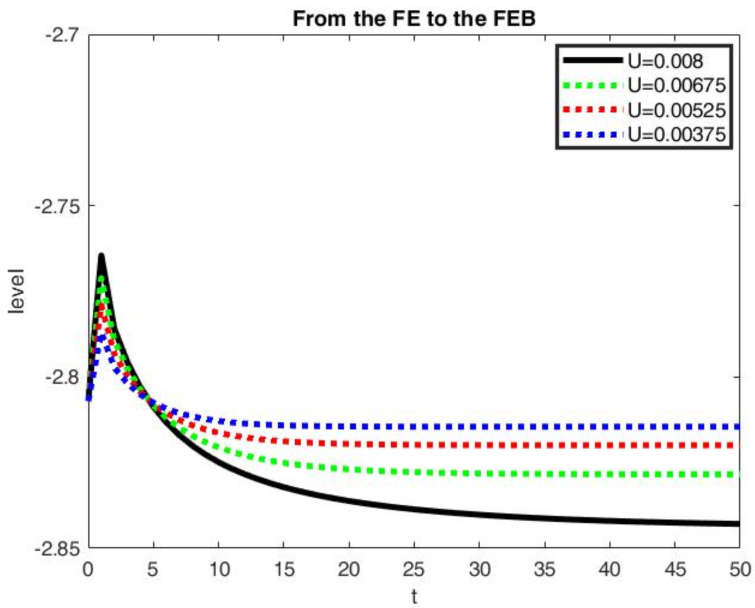


Figure 9: Transitional dynamics

the old (and aggregate) bubble gets smaller in steady state, and converges earlier. As a result, the pattern of interest rates is smoother in Panel A (upper two rows), and the decrease (increase) in the consumption of young (old) households is less pronounced along the transition and at the steady state (bottom row). However, bubbles still redistribute consumption from the young generation to the old one without affecting output as for the benchmark calibration.

In Panel B, the flattening of interest rates' pattern affects output (upper two rows), which, in turn, influences the allocation of consumption (bottom row). For smaller bubbles, the natural interest rate increases slowly, taking the nominal rate out of the ZLB after some periods and not immediately. Hence, output converges gradually at the potential level, and the initial peak in the middle-age consumption vanishes, being replaced by an upward adjustment to the higher steady state level. Middle-aged households can also raise the credit supply less, which lowers the initial peak in the consumption of young households. Notwithstanding, the decline in young-age consumption is less severe, from the second period onward, due to the smaller bubbles absorbing a lower share of saving/credit supply. Overall, the pattern of consumption in young and middle age changes slightly, and the increase in old-age consumption is less intense compared to the benchmark, but the sign of the variation in the generations' steady state consumption is the same, except for young-age consumption when $U = 0.00375$. In this case, young-age consumption rises slowly to a higher steady state level, instead of decreasing after a peak, because it follows the gradual increase in income (and credit supply) of middle-aged households.

As the effect of the bubble on the intergenerational allocation of consumption, in and out of the ZLB, does not change for alternative calibrations of U, the same happens to welfare as measured by (27). As shown in Figure 10, the bubble is always welfare-reducing along the transition from the FE to the FEB equilibrium (upper panel), while it is welfare-enhancing if the economy transitions from the LIR to the FEB equilibrium (bottom panel). The only difference across the calibrations of U is a flatter pattern of welfare, reflecting that of the generations' consumption, for a smaller new bubble.



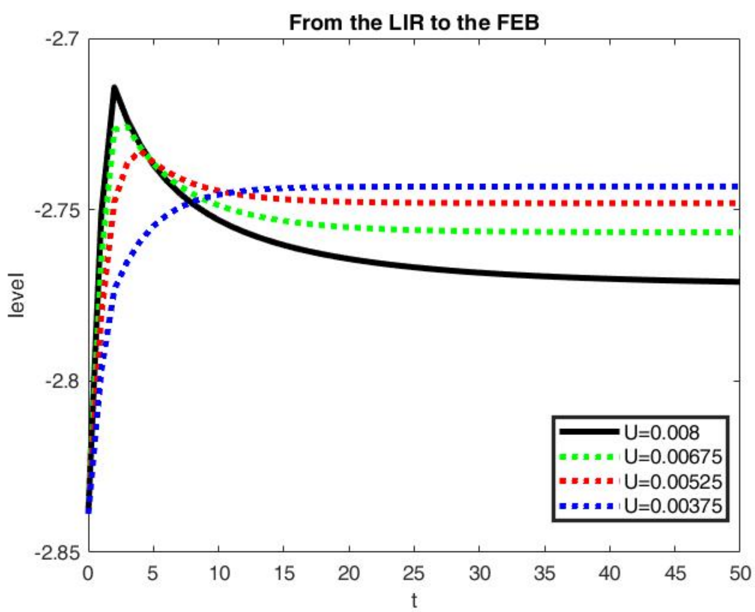


Figure 10: Welfare along the transition

C.1.2 Inflation target

I study the transition from the LIR to the FEB equilibrium by setting $\Pi^* = 1.02$ and $\gamma = 0.99$. As in Appendix C.1.1, different values of U are calibrated to check the robustness of the results. The remaining parameters are the same as the benchmark calibration in Section 4.2 (see Appendix B.4), except for g, which is 0.005 (instead of 0.016).

On the one hand, this alternative calibration of Π^* and γ allows for positive, below the target inflation in the LIR equilibrium ($\Pi = 1.01$ or 1% inflation), consistently with the recent ZLB episodes in the advanced countries. On the other hand, and more importantly, for a positive inflation target, a strictly non-negative natural interest rate is no longer necessary to leave the ZLB (condition 26) because condition (25) does not hold for $r^f < 0$ and $\Pi^* > 1$ if

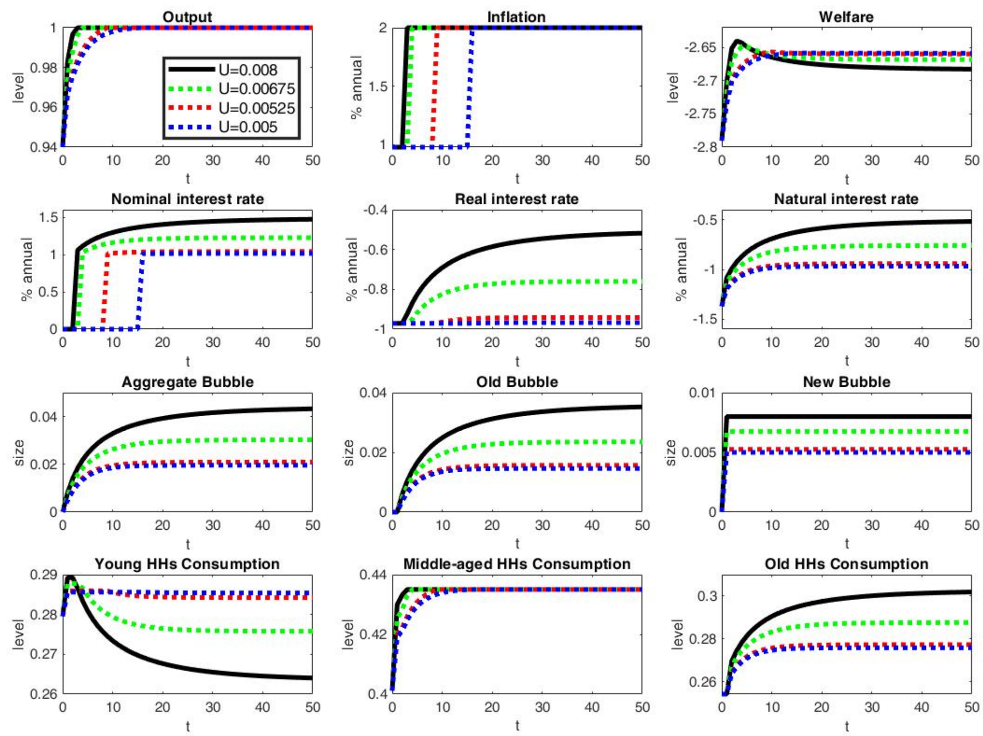
$$\frac{1}{\Pi^*} < 1 + r^f \leq 1.$$

Hence, the transition from the LIR to the FEB equilibrium can happen even if bubbles push the natural interest rate up, but it remains negative as long as the positive inflation target is high enough. This is precisely the case under investigation here.

I plot the transitional dynamics with this alternative calibration in Figure 11.[33] It is evident that nothing changes substantially with respect to Panel B of Figure 4, and smaller (new and old) bubbles flatten the pattern of the variables, as in Appendix C.1.1 (bottom panel of Figure 9). Although the natural interest rate rises without reaching a positive value, its increase shrinks the gap with the real interest rate, stimulating output. At some point, the natural rate is sufficiently high, though negative, that the economy escapes from the ZLB, with output reaching the potential and inflation hitting the target (the upper two rows). On the other hand, the overall (direct and indirect) effect of bubbles on the allocation of consumption across generations and welfare is the same as in Section 4.2 (bottom row and far-right panel in the top row). Middle-age and young-age consumption increase along the transition to the FEB equilibrium, while the consumption in young age decreases, except for minimal values of the new bubble's size. Overall, the effect of this intergenerational redistribution is positive, with higher lifetime utility in the new steady state.

[33]The same calibrations of U in Appendix C.1.1 correspond now to negative values for r^f in steady state due to the lower g. I replace the lowest calibration of U in Appendix C.1.1, 0.00375, with 0.005 because otherwise the economy would not leave the ZLB with a lower g. For any calibration of U and the corresponding r^f, the condition for the uniqueness of the FEB equilibrium (see footnote 13), $\Pi^* > 1/\gamma(1 + r^f)$, is satisfied.

Figure 11: Transitional dynamics from the LIR to the FEB



D Extensions

D.1 Capital and Endogenous Fundamental Collateral

Households. I assume zero population growth ($g = 0$) and the size of the generations is constant to one. The household's problem is unchanged with respect to Section 3 except for the choice of capital at young age and taxes levied on income in middle age:[34]

$$\max_{C^m_{t+1}, C^o_{t+2}, K_t, Q^B_{t+1|t+1-j}} E_t \left\{ \ln C^y_t + \beta \ln C^m_{t+1} + \beta^2 \ln C^o_{t+2} \right\}$$

s.t.

$$C^y_t + K_t = B^y_t = \frac{\phi E_t Y_{t+1} + \delta E_t P^B_{t+1|t+1}}{(1+r_t)}$$

$$C^m_{t+1} = (1-\tau) Y_{t+1} + \delta P^B_{t+1|t+1} - (1+r_t) B^y_t - B^m_{t+1} - \sum_{j=0}^{\infty} P^B_{t+1|t+1-j} Q^B_{t+1|t+1-j} \quad (46)$$

$$C^o_{t+2} = (1+r_{t+1}) B^m_{t+1} + (1-\delta) \sum_{j=0}^{\infty} P^B_{t+2|t+1-j} Q^B_{t+1|t+1-j}, \quad (47)$$

where $\phi \in (0, 1-\tau)$ and $Y_{t+1} = \frac{W_{t+1}}{P_{t+1}} L_{t+1} + r^k_{t+1} K_t$. The optimality conditions of the household's problem are (5), (6) and (30).

Firms. The maximization problem of firms is given by

$$\max_{K_t, L_t} Y_t - \frac{W_t}{P_t} L_t - r^k_t K_{t-1}$$

s.t.

$$Y_t = K^{1-\alpha}_{t-1} L^\alpha_t. \quad (48)$$

Hence, labor and capital are remunerated at their marginal productivity:

$$\frac{W_t}{P_t} = \alpha K^{1-\alpha}_{t-1} L^{\alpha-1}_t = \alpha \frac{Y_t}{L_t} \quad (49)$$

$$r^k_t = (1-\alpha) K^{-\alpha}_{t-1} L^\alpha_t = (1-\alpha) \frac{Y_t}{K_{t-1}}. \quad (50)$$

[34] Young households born at time $t-1$ are borrowing constrained for

$$\phi < \frac{1}{1+\beta(1+\beta)} \left\{ (1-\tau) + \beta(1+\beta) \left[(1+r_{t-1}) \frac{K_{t-1}}{Y_t} - \frac{\delta P^B_{t|t}}{Y_t} \right] \right\}.$$

The introduction of capital slightly changes the shape of the DNWR, which becomes

$$W_t = \max\left(\gamma \Pi^* W_{t-1}, P_t \alpha K_{t-1}^{1-\alpha} \bar{L}^{\alpha-1}\right).$$

Monetary and fiscal policy. The behavior of the central bank, in charge of monetary policy, is still summarized by the interest rate rule (11). There also exists a government in charge of fiscal policy. It finances, through proportional taxes on income and public debt B_t^g in the form of one-period bonds, government spending G_t and debt service, as described by the government budget constraint:

$$\tau Y_t + B_t^g = G_t + (1 + r_{t-1}) B_{t-1}^g, \tag{51}$$

where τ is the tax rate. The government keeps the public debt constant at the level $\bar{B}^g$, and thus government spending adjusts to satisfy equation (51).

Credit and bubbles markets. The market for bubbles is the same as in Section 3.[35] Instead, the innovation in the borrowing constraint alters credit demand and supply. The credit demand, in the case of perfect foresight, becomes

$$D_t^c = \frac{\phi Y_{t+1} + U_{t+1}}{(1 + r_t)} + \bar{B}^g, \tag{52}$$

while the credit supply takes the shape

$$S_t^c = B_t^m = \frac{\beta}{1+\beta}\left[(1 - \tau - \phi) Y_t - U_t - B_t\right] - \frac{1}{1+\beta}(U_t + B_t). \tag{53}$$

Since the government requests resources from the credit market, along with borrowing-constrained young households, credit demand consists of the fixed amount of public debt and the (present) value of fundamental and bubbly collaterals. On the other hand, bubbles still affect the credit supply by serving as a store of value and a collateral, as described in Section 3. Finally, the equilibrium real interest rate equating credit demand and supply, $D_t^c = S_t^c$, is

$$1 + r_t = \frac{(1+\beta)(\phi Y_{t+1} + U_{t+1})}{\beta\left[(1 - \tau - \phi) Y_t - U_t - B_t\right] - U_t - B_t - (1+\beta)\bar{B}^g}, \tag{54}$$

and it corresponds to the natural interest rate for $Y_t = Y_t^f$.

[35] However, the condition for the existence of a bubbly steady state is now $\phi < \frac{\beta}{1+\beta}(1 - \tau - \phi) - (1+\beta)\frac{\bar{B}^g}{Y}$, implying $1 + r < 1$ in the bubbleless economy with no population growth ($g = 0$).

Aggregate supply and demand. The AS coincides with potential output, $Y_t^f = K_{t-1}^{1-\alpha}\bar{L}^{\alpha}$, when the DNWR is not at work, while it is

$$Y_t = \frac{\gamma\Pi^*}{\Pi_t}\left(\frac{L_t}{\bar{L}}\right)Y^f \tag{55}$$

in case of binding DNWR. The general shape of the AD is

$$Y_t = \frac{1}{1-\tau-\phi}\left(\frac{1+\beta}{\beta}\right)\left[U_t + B_t + \bar{B}^g + \left(\frac{\phi Y_{t+1} + U_{t+1}}{1+r_t}\right)\right], \tag{56}$$

where $1 + r_t$ is given by (31) if the ZLB does not bind, and $1 + r_t = \frac{1}{E_t\Pi_{t+1}}$ at the ZLB.

D.2 Unleveraged vs Leveraged Bubble

D.2.1 Unleveraged bubble

For $\phi = 0$, young households cannot collateralize the bubble and there is no default risk, $\xi = h = 0$. Given $Q^{B,y} = 0$, the budget constraint in young age is $C^y = B^y = D/(1+r)$. The household faces, instead, the following constraints in middle and old age:

$$C^m = \begin{cases} Y - B^m - (1+r)\,B^y & bubble \quad bursting \\ Y - P^B Q^{B,m} - B^m - (1+r)\,B^y & bubble \quad survival, \end{cases}$$

$$C^o = \begin{cases} (1+r)\,B^m & bubble \quad bursting \\ (1+r)\,B^m + P^B Q^{B,m} & bubble \quad survival. \end{cases}$$

Denote consumption levels in case of bubble bursting as $C^{m,b}$ and $C^{o,b}$ and those in case of bubble survival as $C^{m,s}$ and $C^{o,s}$. The optimality conditions of the household's problem are

$$\frac{1}{C^{m,s}} = \beta\,(1+r)\left[\rho\frac{1}{C^{o,b}} + (1-\rho)\,\frac{1}{C^{o,s}}\right]$$

$$\frac{1}{C^{m,s}} = \beta\,(1-\rho)\,\frac{1}{C^{o,s}},$$

which express the optimal choice of bonds and bubbles from middle-aged households. We can derive the price of the unleveraged bubble and the corresponding real interest rate in a fully unleveraged bubbly equilibrium, in which $Q_t^{B,m} = 1$, by using these optimality conditions, the borrowing constraint, $B^y = D/(1+r)$, and the equilibrium condition for the credit market (17):

$$P^{BU} = (1-\rho)\frac{\beta}{1+\beta}(Y-D) - D$$
$$(1+r) = \frac{(1+\beta)D}{\rho\beta(Y-D)+(1+\beta)D} < 1.$$

In the unleveraged bubbly economy, the bubble can be used only as a store of value, and the corresponding AD is

$$Y = D + \left(\frac{1+\beta}{\beta}\right)P^{BU} + \left(\frac{1+\beta}{\beta}\right)\left(\frac{\Pi^*}{\Pi}\right)^{\phi_\pi - 1}\frac{D}{(1+r^f)}, \tag{57}$$

if the ZLB is not binding, while it is

$$Y = D + \left(\frac{1+\beta}{\beta}\right)P^{BU} + \left(\frac{1+\beta}{\beta}\right)\Pi D \tag{58}$$

if the ZLB is binding.

D.2.2 Leveraged bubble

Given $\phi = 1$, the budget constraint in young age is $C^y = B^y - P^B Q^{B,y}$ with $B^y = (D + P^B Q^{B,y})/(1+r)$, while the budget constraint in middle age becomes

$$C^m = \begin{cases} Y - B^m - D & bubble \quad bursting \\ Y + P^B Q^{B,y} - B^m - (1+r)B & bubble \quad survival. \end{cases}$$

The young households can only buy the bubble ($Q^{B,y} = 1$ and $Q^{B,m} = 0$). Hence, the budget constraint in old age is $C^o = (1-h)(1+r)B^m$, and middle-aged households invest all their savings in bonds, with their optimal choice expressed by the Euler equation (5). The other optimality conditions of the representative household are

$$\frac{1}{C^y} = \beta(1-\rho)\frac{1}{C^{m,s}} + \lambda^D$$

$$\lambda^D = \frac{1}{C^y(1+r)} - \beta(1-\rho)\frac{1}{C^{m,s}} > 0,$$

where $C^{m,s}$ denotes the middle-age consumption in case of bubble survival. λ^D is the lagrangian multiplier associated with the borrowing constraint, and it has to be positive because the household is borrowing constrained in young age. Combining these two conditions, the Euler equation (5) and the credit market clearing condition (17) obtains

$$P^{BL} = \frac{\beta}{1+\beta}(Y - D) - D$$

and

$$1 + r = 1 + r^f = 1.$$

The bubble plays the unique role of collateral in the leveraged bubbly economy. Hence, the AD is

$$Y = D + \left(\frac{1+\beta}{\beta}\right) P^{BL} + \left(\frac{1+\beta}{\beta}\right)\left(\frac{\Pi^*}{\Pi}\right)^{\phi_\pi - 1}\left(\frac{D + P^{BL}}{1+r^f}\right), \tag{59}$$

far away from the ZLB, and it becomes

$$Y = D + \left(\frac{1+\beta}{\beta}\right) P^{BL} + \left(\frac{1+\beta}{\beta}\right) \Pi \left(D + P^{BL}\right) \tag{60}$$

if the ZLB binds.